Preliminary evidence for a stable 2-sphere in the Yang-Mills flow for $SU(3)$ gauge fields on S^4

Addendum July 6, 2009

Daniel Friedan

Department of Physics & Astronomy, Rutgers the State University of New Jersey
and
Natural Science Institute, University of Iceland

At the workshop *Geometric Flows in Mathematics and Theoretical Physics*, Pisa, June 24, 2009, I described some an ongoing attempt to understand the long time behavior of the Yang-Mills flow acting on topologically non-trivial 2-spheres of $SU(3)$ connections on S^4 [1].

This was very much a report on work in progress. I sketched a half-baked plan for calculating the Yang-Mills flow near configurations of an instanton/anti-instanton pair, both asymptotically small. After further thought, I think I can half-bake the plan a bit more.

The goal is to find a stable 2-sphere by finding a two-dimensional unstable manifold of small instanton/small anti-instanton pairs. Computer calculations suggested that such an unstable manifold might well exist.

The instanton and the anti-instanton are separately stable under the flow, so the flow will concentrate on the zero-mode space: the moduli space of the small-instanton/small-anti-instanton pair. The zero-modes are the locations and sizes of the instanton and the anti-instanton, and the internal moduli that describe their relative orientation within $SU(3)$.

Plan of calculation

Write ξ^I for the moduli. Let $A_\pm(\xi)$ be the self-dual solution on one hemisphere and the anti-self-dual solution on the other hemisphere. The two solutions do not quite fit together at the boundaries of the hemispheres, so they must be corrected slightly

$$A(\xi) = A_\pm(\xi) + \delta A_\pm(\xi) \tag{1}$$

to get a connection that is continuous and differentiable at the common boundary of the two hemispheres.

This family of connections, $A(\xi)$, is supposed to be closed under the Y-M flow. That is, there is supposed to be a flow $\xi(t)$ on the zero-modes such that $A(\xi(t))$ is the Y-M flow:

$$\frac{d}{dt}A(\xi(t)) = \dot{\xi}^I \frac{\partial A}{\partial \xi_I} = *d*F(\xi(t))\,. \tag{2}$$

It should be possible to solve perturbatively for the $\delta A_\pm(\xi)$, because the flow comes to a stop when the sizes vanish.

The plan is to solve for the $\delta A_\pm(\xi)$ separately on each hemisphere, then choose among all possible solutions the pair that make $A(\xi)$ continuous and differentiable at the common boundary of the hemispheres. The choice should be unique.

The linearized equation is

$$\dot{\xi}^I \frac{\partial A_\pm}{\partial \xi_I} = \Box_{YM}\delta A_\pm(\xi) \tag{3}$$

On each hemisphere, choose specific solutions $B_{I\pm}$ of

$$\Box_{YM} B_{I\pm} = \frac{\partial A_\pm}{\partial \xi_I} \tag{4}$$

The most general solution to the linearized equation on the separate hemispheres is

$$\delta A_\pm = \dot{\xi}^I B_{I\pm} + N_\pm \tag{5}$$

for some $N_\pm$ satifying

$$\Box_{YM} N_\pm = 0\,. \qquad (6)$$

Now require that $A_\pm + \delta A_\pm$ be continuous and differentiable at the common boundary, with $\delta A_\pm$ given by equation 5. This should determine the velocities $\dot{\xi}^I$ uniquely.

For this to work, the space of boundary values of solutions of $\Box_{YM} N_\pm = 0$ in each hemisphere must have have co-dimension N in the space of boundary conditions (N being the number of zero-modes in each hemisphere).

Atiyah, Hitchen and Singer [2] showed that there exist irreducible $SU(n)$ instantons on S^4 of degree k iff $k \geq \frac{1}{2}n$. So all $SU(3)$ instantons of degree $k = 1$ on S^4 live in an $SU(2)$ subgroup. The linearized equations in each hemisphere are thus in the background of a small $SU(2)$ instanton. There are explicit formulas for Green's functions in the $SU(2)$ instanton background, so the calculation is probably doable.

Speculation

One might guess that the instanton and anti-instanton have to be lined up perfectly in order to merge together so that their topological charges can eventually cancel. It is tempting to speculate that there is only a single flow line along which the small-instanton and the small-anti-instanton would grow larger, eventually merging to reach the flat connection.

The stable 2-sphere would then lie entirely within the moduli space of zero-modes, except for an infinitesimally thin tube flowing from the S-pole of the 2-sphere down to the flat connection.

The internal moduli space for a small $SU(2)$ instanton and a small $SU(2)$ anti-instanton in $SU(3)$ appears to be essentially $\mathbb{CP}^2$, and $\pi_2\mathbb{CP}^2 = \mathbb{Z}$, so this scenario might be feasible.

SU(2)

I'm still puzzled by the $SU(2)$ flow. The only model I've seen for the generator of $\pi_5 SU(2)$ is the map $S(S(H)) \circ S(H) : S^5 \to S^3$, where $H : S^3 \to S^2$ is the Hopf map, $S(H) : S^4 \to S^3$ its suspension, and $S(S(H)) : S^5 \to S^4$ its double suspension. This gives the non-trivial $SU(2)$ bundle over S^6, but I have not been able to find a useful construction. I have no idea what a stable 2-sphere for the $SU(2)$ flow might look like.

[1] D. Friedan, *Preliminary evidence for a stable 2-sphere ...*, Pisa, June 24, 2009, http://www.crm.sns.it/download/corsi/2121/Friedan.pdf or http://www.physics.rutgers.edu/pages/friedan/talks/flows/Friedan_2009.06.24_Pisa.pdf.

[2] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-Duality in Four-Dimensional Riemannian Geometry*, Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, Vol. 362, No. 1711 (Sep. 12, 1978), pp. 425-461.