A conjecture on the Ricci flow

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April 16, 2008

I'll state the conjecture in a very specific form. There are several obvious avenues of generalization.

Has this conjecture already been studied? Is it foolish? If not, I wonder what are the chances of proving it? Could it provide a useful method of constructing Ricci solitons and families of flows between them?

Let $\tilde{\mathcal{M}}_4$ be the space of AE metrics on $\mathbb{R}^4$ (going to a fixed euclidean metric g_E at infinity), modulo diffeomorphisms of $\mathbb{R}^4$ that go to the identity map at infinity.

Let $\mathcal{M}_4$ be the connected component of $\tilde{\mathcal{M}}_4$ that contains the flat euclidean metric g_E.

$$\pi_2(\mathcal{M}_4) = \pi_1(\mathrm{Diff}(S^4)) = \pi_5(S^4) = \mathbb{Z}/2\mathbb{Z} \tag{1}$$

(To resolve the singularity at the euclidean metric g_E, we have to do something like adjoining a frame at infinity to the space of metrics, before dividing by the diffeomorphism group.)

Choose an arbitrary representative in the nontrivial homotopy class

$$G_0 : S^2 \to \mathcal{M}_4 \qquad G_0(s) = g_E \qquad s = \text{the south pole in } S^2\,. \tag{2}$$

Run the Ricci flow pointwise on this 2-sphere of metrics

$$G_t(x) = G(x)_t \qquad x \in S^2 \qquad t \geq 0. \tag{3}$$

Conjecture

Under the Ricci flow, any 2-sphere of metrics in the nontrivial homotopy class ends as a trapped stable 2-sphere in the same homotopy class:

$$\lim_{t\to\infty} G_t = G_\infty \mod \mathrm{Diff}(S^2) \tag{4}$$

where

1. $G_\infty(s) = g_E$, the euclidean metric.
2. $G_\infty(n)$ is a nontrivial fixed point (Ricci soliton), n being the north pole in S^2.
3. $G_\infty(S^2)$ is the unstable manifold of $G_\infty(n)$.
4. $G_\infty : S^2 \to \mathcal{M}_4$ represents the nontrivial homotopy class.

The euclidean group of $\mathbb{R}^4$ acts on the manifold of all such trapped stable surfaces. Are they all equivalent under the euclidean group?

I'm also interested in the analogous conjecture for the Yang-Mills flow.

Let $\mathcal{M}_{\mathrm{YM},4}$ to be the asymptotically flat $SU(3)$ gauge fields on euclidean $\mathbb{R}^4$, modulo gauge equivalence, the connected component of the flat gauge field.

$$\pi_2(\mathcal{M}_{\mathrm{YM},4}) = \pi_5(SU(3)) = \mathbb{Z}/2\mathbb{Z} \tag{5}$$

Run the Yang-Mills flow on a representative G_0 of the nontrivial homotopy class. The conjecture is that G_0 flows to a trapped stable 2-sphere, which is the unstable manifold of a solution of the Yang-Mills equation. The conformal group of $\mathbb{R}^4$ acts on the space of trapped stable 2-spheres.