# Quantum field theories of $(n-1)$-dimensional extended objects in $2n$-dimensional space-time manifolds as 2d quantum field theories on "quasi Riemann surfaces" of integral $(n-1)$-currents

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# Abstract

This is a project to develop a wide expanse of new quantum field theories in $2n$-dimensional space-time manifolds.

For every 2d qft, there is to be a qft of extended objects in every $2n$-dimensional space-time manifold $M$.

The quantum fields live on "quasi Riemann surfaces", which are certain spaces of integral $(n-1)$-currents in $M$. The notion of *integral current* comes from Geometric Measure Theory.

The quasi Riemann surfaces are complete metric spaces with analytic properties precisely analogous to Riemann surfaces.

The new qfts are to be constructed on the quasi Riemann surfaces exactly as 2d qfts are constructed on ordinary Riemann surfaces.

Local fields in space-time will be obtained by restricting to small extended objects.

# References

DF, *Quantum field theories of extended objects*
arXiv:1605.03279 [hep-th]
and references therein

DF, *Quasi Riemann surfaces*
in preparation

# The line of thought

1. Re-write the abelian $U(1)$ gauge theory of the free $n$-form in $2n$ dimensions as a 2d cft, the $c{=}1$ gaussian model, on certain spaces of integral $(n{-}1)$-currents.
2. These "quasi Riemann surfaces" have precisely the analytic properties of ordinary Riemann surfaces.
3. Recall ancient history — all of 2d cft is constructed from the 2d gaussian model, and all of non-conformal 2d qft.
4. Envision extending all the constructions of 2d cft from ordinary Riemann surfaces to quasi Riemann surfaces,
   thereby constructing a cft of $(n{-}1)$-dimensional extended objects in $2n$ dimensions for every 2d cft.

   Then envision extending all the constructions of non-conformal 2d qft to obtain a non-conformal qft of extended objects for each non-conformal 2d qft.

# Marketing

We know a huge variety of 2d cft's and qft's. We can do explicit, exact calculations in very many of them.

The plan is to make a cft of extended objects in $2n$ dimensions for for every one of these 2d cft's (and a qft from every 2d qft).

They are to be constructed by a completely new technique for making quantum field theories in $2n$ dimensions, which at the same time incorporates all the techniques of 2d cft and 2d qft.

Some of these will equal known $2n$-dimensional qft's. Almost all of them are likely to be new. In the former case, there will be new techniques for calculating in known qft's. In the latter, there will be new qft's.

So I'm offering a new playground of qft in $2n$ dimensions — an opportunity to develop a new technology for doing qft — a rich collection of foundational technical problems to solve.

# Marketing (2)

I do not have any physical applications in mind. But there are indications that some of these theories will have nonabelian local gauge symmetry in $2n{=}4$ dimensions.

# The manifold $M$

The euclidean space-time $M$ is

- a manifold of dimension $2n$
- oriented
- with a conformal structure (euclidean signature)
- compact and without boundary (for simplicity)

The basic example is $M = S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$.

The Hodge $*$-operator acting on $n$-forms is conformally invariant

$$(*\omega)_{\nu_1 \cdots \nu_n}(x) = \omega_{\mu_1 \cdots \mu_n}(x) \, \frac{1}{n!} \, \epsilon^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_n}(x)$$

and satisfies

$$*^2 = (-1)^n$$

Nothing else of the conformal structure will be used.

(Better to say "*space with Hodge-$*$ in the middle dimension*" instead of "*manifold with conformal structure*".)

# The free $n$-form classical field theory

$F(x)$ is an $n$-form on $M$.

In $2n{=}4$ dimensions, $F(x)$ is the $2$-form electromagnetic field.

The classical equations of motion

$$dF = 0, \qquad d(*F) = 0$$

allow deriving $(n{-}1)$-form gauge potentials (locally, at least)

$$dA = F, \qquad dA^* = *F$$

defined up to $(n{-}2)$-form gauge symmetries $f$, $f^*$

$$A \to A + df, \qquad A^* \to A^* + df^*$$

(will quantize tomorrow)

# Extended objects in the free $n$-form theory

The basic fields of extended objects are

$$V_{p,p^*}(\xi) = e^{ip\int_\xi A + ip^*\int_\xi A^*}$$

An extended object $\xi$ is some kind of "subspace in $M$" over which the $(n{-}1)$-forms $A$ and $A^*$ can be integrated.

Example in $2n{=}4$ dimensions: $\xi$ = the 1-dimensional euclidean world-line of a dyon of electric charge $p$ and magnetic charge $p^*$.

What exactly are the $(n{-}1)$-dimensional extended objects $\xi$?

They should include the $(n{-}1)$-submanifolds. And what else?

I emphasize that we want the extended objects for the free $n$-form. We are not concerned with extended objects for any other theory.

# $U(1) \times U(1)$ gauge symmetry

$A$ and $A^*$ are gauge potentials for the compact gauge group $U(1)$

$(A_1, A_1^*)$ is equivalent to $(A_2, A_2^*)$ if, for all $\xi$,

$$\int_\xi (A_1 - A_2) \in 2\pi R\mathbb{Z} \quad \text{and} \quad \int_\xi (A_1^* - A_2^*) \in 2\pi R^*\mathbb{Z}$$

which is to say $\int_\xi A \in S^1_R$ and $\int_\xi A^* \in S^1_{R^*}$

So $e^{ip\int_\xi A + ip^*\int_\xi A^*}$ lies in the charge lattice $p, p^* \in \mathbb{Z}/R \times \mathbb{Z}/R^*$.

The $U(1)$ condition is consistent with taking *integer* linear combinations of the $\xi$, but *not* real linear combinations.

So the extended objects form an abelian group.

Compare lattice $U(1)$ gauge theory, where the space of extended objects is the abelian group generated by the $(n-1)$-placquettes.

Abelian-ness of the $n$-form gauge theory is essential.

# Extended objects as currents

I will use '*current*' in the mathematical sense, specifically in the sense of Geometric Measure Theory (GMT).

The $k$-currents are the distributions (linear functionals) on $k$-forms

$$\omega \mapsto \int_\xi \omega = \int_M \omega_{\mu_1 \cdots \mu_k}(x)\, \xi^{\mu_1 \cdots \mu_k}(x) d^d x$$

Certainly, an extended object $\xi$ is *some* kind of $(n-1)$-current, something over which we can integrate $(n-1)$-forms $A$, $A^*$.

We want $k$-currents that are delta-functions on $k$-dimensional objects in $M$. (Think lines of current as 1-dimensional objects.)

The oriented $k$-simplex $\Delta^k$ is the basic $k$-dimensional object:
(0) point, (1) line interval, (2) triangle, (3) tetrahedron, ...

An oriented $k$-simplex in $M$ gives a $k$-current

$$\sigma\colon \Delta^k \to M\,, \qquad \int_{[\sigma]} \omega = \int_{\Delta^k} \sigma^* \omega$$

$[\sigma]$ is the characteristic $\delta$-function localized on $\sigma(\Delta^k) \subset M$.

$\mathcal{D}_k^{sing}(M) = \big\{$integer linear combinations $\sum_i m_i[\sigma_i]\big\}$ is the abelian group of *singular* $k$-currents. It includes the $k$-submanifolds.

The singular $k$-current is the physical object in $M$, the linear functional on $k$-forms, independent of how it is made out of $k$-simplices as $\sum_i m_i[\sigma_i]$.

# The boundary operator on currents

The boundary operator on currents is defined as the dual of the exterior derivative on forms (Stokes' theorem by definition)

$$\mathcal{D}_k^{distr}(M) \xrightarrow{\partial} \mathcal{D}_{k-1}^{distr}(M)\,, \qquad \int_{\partial\xi} \omega = \int_\xi d\omega$$

$$(\partial\xi)^{\mu_2\cdots\mu_k}(x) = -\partial_{\mu_1}\xi^{\mu_1\mu_2\cdots\mu_k}(x)$$

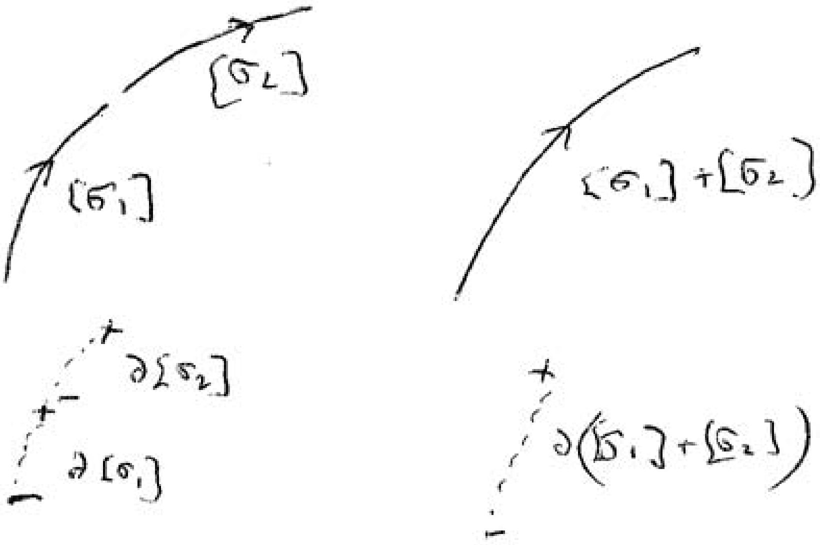
$$d^2 = 0 \quad \Longrightarrow \quad \partial^2 = 0$$

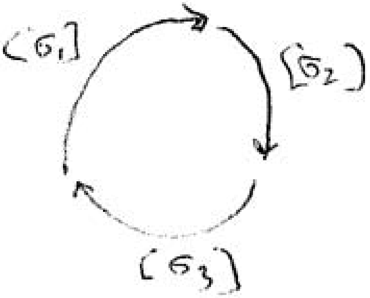
The boundary of a singular current is a singular current

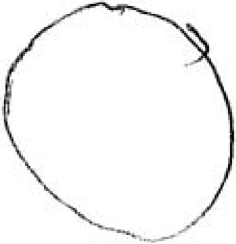
$$\mathcal{D}_k^{sing}(M) \xrightarrow{\partial} \mathcal{D}_{k-1}^{sing}(M)$$

On submanifolds, $\partial$ is the usual boundary operator.

# Figure 1





$$[\sigma_1] + [\sigma_2] + [\sigma_3] = 3$$

$$\partial 3 = 0$$

GMT puts a certain metric on $\mathcal{D}_k^{sing}(M)$ and then takes the metric completion to get the *integral* currents

$$\mathcal{D}_k^{sing}(M) \subset \mathcal{D}_k^{int}(M) \subset \mathcal{D}_k^{distr}(M)$$

$$\mathcal{D}_k^{int}(M) \xrightarrow{\partial} \mathcal{D}_{k-1}^{int}(M)$$

The additional currents, the limits of Cauchy sequences, are fractal.

Take the space of extended objects to be $\mathcal{D}_{n-1}^{int}(M)$.

We go to the metric completion in order to get a mathematically well-defined calculus of differential forms on the space of extended objects (about which more a bit later).

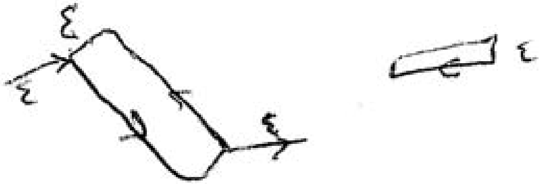
# The flat metric

$$\mathrm{dist}(\xi_1, \xi_2) = \|\xi_1 - \xi_2\|_{\mathit{flat}}$$

$$\|\xi\|_{\mathit{flat}} = \inf_{\xi'} \left[ (k+1)\text{-volume}(\xi') + k\text{-volume}(\xi - \partial\xi') \right]$$

The flat norm $\|\xi\|_{\mathit{flat}}$ measures how easy it is to deform $\xi \to 0$.

$\xi =$




$\|\xi\|_{flat} = O(\varepsilon)$

# Recall the 2d gaussian model

the 2d gaussian model = the free 1-form in $2n{=}2$ dimensions

$F$, $A$, $A^*$ are written $j$, a 1-form, and $\phi$, $\phi^*$, 0-forms (scalar fields).

$$j = d\phi, \qquad *j = d\phi^*, \quad \phi(x) \in S^1_R\,, \quad \phi^*(x) \in S^1_{R^*}$$

The vertex operators (which live at points $x$) are

$$V_{p,p^*}(x) = e^{ip\phi(x)+ip^*\phi^*(x)}, \qquad p, p^* \in \mathbb{Z}/R \times \mathbb{Z}/R^*$$

The global $U(1)\times U(1)$ symmetry is

$$\phi \to \phi + a, \quad \phi^* \to \phi + a^*, \qquad V_{p,p^*}(x) \to V_{p,p^*}(x)\, e^{ipa+ip^*a^*}$$

(tomorrow: complex coordinate $z = x^1 + ix^2$, chiral 1-forms, quantization, $RR^* = 1$ )

# The free $n$-form as 2d gaussian model

Define scalar fields on $\mathcal{D}^{int}_{n-1}(M)$

$$\phi(\xi) = \int_\xi A \qquad \phi^*(\xi) = \int_\xi A^*$$

The extended object fields now take the form

$$V_{p,p^*}(\xi) = e^{ip\phi(\xi)+ip^*\phi^*(\xi)}$$

Under a gauge transformation $A \to A + df\,,\ \ A^* \to A^* + df^*$

$$\phi(\xi) \to \phi(\xi) + \int_\xi df\,, \qquad \phi^*(\xi) \to \phi^*(\xi) + \int_\xi df^*$$

$$\int_\xi df = \int_{\partial\xi} f = a(\partial\xi)\,, \qquad \int_\xi df^* = \int_{\partial\xi} f^* = a^*(\partial\xi)$$

so

$$V_{p,p^*}(\xi) \;\longrightarrow\; V_{p,p^*}(\xi)\; e^{ipa(\partial\xi)+ip^*a^*(\partial\xi)}$$

# The free $n$-form as 2d gaussian model (2)

If we fix an $(n-2)$-boundary $\partial\xi_0$ and consider only the $\xi$ that have the same boundary $\partial\xi = \partial\xi_0$,

$$\mathcal{D}^{int}_{n-1}(M)_{\partial\xi_0} = \left\{\xi \in \mathcal{D}^{int}_{n-1}(M) \colon\ \partial\xi = \partial\xi_0\right\}$$

then the generators are just two numbers, independent of $\xi$,

$$a(\partial\xi) = a(\partial\xi_0), \qquad a^*(\partial\xi) = a^*(\partial\xi_0)$$

The gauge symmetry becomes a global $U(1)\times U(1)$

$$V_{p,p^*}(\xi) \to V_{p,p^*}(\xi)\, e^{ipa(\partial\xi_0)+ip^*a^*(\partial\xi_0)}$$

It begins to look like there could be a 2d gaussian model on each $\mathcal{D}^{int}_{n-1}(M)_{\partial\xi_0}$.

# The bundle $\mathcal{Q}(M) \to \mathcal{PB}(M)$

$\mathcal{D}^{int}_{n-1}(M)_{m\partial\xi_0}$ sees the same $U(1) \times U(1)$ generators, since

$$a(m\partial\xi_0) = \int_{m\partial\xi_0} f = ma(\partial\xi_0)\,, \quad a^*(m\partial\xi_0) = \int_{m\partial\xi_0} f^* = ma^*(\partial\xi_0)$$

so combine them (in a disconnected sum)

$$\mathcal{D}^{int}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} = \bigoplus_{m\in\mathbb{Z}} \mathcal{D}^{int}_{n-1}(M)_{m\partial\xi_0} = \left\{\xi \in \mathcal{D}^{int}_{n-1}(M) \colon \partial\xi \in \mathbb{Z}\partial\xi_0\right\}$$

These will be the "quasi Riemann surfaces". Each is an abelian group and a complete metric space. They are parametrized by the "integer lines"

$$\mathcal{PB}(M) = \left\{\text{maximal } \mathbb{Z}\partial\xi_0 \subset \partial\mathcal{D}^{int}_{n-1}(M)\right\}$$

Collectively, they form the bundle of "quasi Riemann surfaces"

$$\mathcal{Q}(M) \to \mathcal{PB}(M)\,, \qquad \mathcal{Q}(M)_{\mathbb{Z}\partial\xi_0} = \mathcal{D}^{int}_{n-1}(M)_{\mathbb{Z}\partial\xi_0}$$

On each fiber $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi_0}$ there will be a 2d gaussian model.

# Three reasons for the metric completion

For brevity, write $\mathcal{Q}$ for any one of the fibers

$$\mathcal{Q} = \mathcal{Q}(M)_{\mathbb{Z}\partial\xi_0} = \mathcal{D}^{int}_{n-1}(M)_{\mathbb{Z}\partial\xi_0}$$

By construction, by the metric completion, $\mathcal{Q}$ is an abelian group and a complete metric space.

(1) GMT gives a construction of integral currents in any complete metric space, so a well-defined calculus of integral $j$-currents in $\mathcal{Q}$,

$$\mathcal{D}^{int}_j(\mathcal{Q}) \xrightarrow{\partial} \mathcal{D}^{int}_{j-1}(\mathcal{Q})$$

from which we can define the dual $j$-forms on $\mathcal{Q}$

$$\Omega_j(\mathcal{Q}) = \mathrm{Hom}(\mathcal{D}^{int}_j(\mathcal{Q}), \mathbb{R})\,, \qquad \Omega_j(\mathcal{Q}) \xrightarrow{d} \Omega_{j+1}(\mathcal{Q})$$

# Three reasons for the metric completion (2)

(2) There are natural maps

$$\Pi_j \colon \mathcal{D}_j^{int}(\mathcal{Q}) \to \mathcal{D}_{j+n-1}^{int}(M)\,, \qquad \Pi_{j-1}\partial = \partial\,\Pi_j$$

Their dual maps pull back forms from $M$ to $\mathcal{Q}$

$$\Pi_j^* \colon \Omega_{j+n-1}(M) \to \Omega_j(\mathcal{Q})\,, \qquad d\,\Pi_{j-1}^* = \Pi_j^* d$$

These come from writing $\Delta^j \times \Delta^k$ as a sum of $(j{+}k)$-simplices.
So a $j$-parameter family of $k$-simplices in $M$ is a $(j{+}k)$-current.
So there are natural maps

$$\Pi_{j,k} \colon \mathcal{D}_j^{int}(\mathcal{D}_k^{int}(M)) \to \mathcal{D}_{j+k}^{int}(M)$$

Setting $k = n - 1$ gives $\Pi_j$.

# Three reasons for the metric completion (3)

(3) A $*$ operator can be defined on 1-forms on $\mathcal{Q}$ that is compatible with the Hodge-$*$ operator on $n$-forms on $M$,

$$* \, \Pi_1^* = \Pi_1^* \, *$$

This is proved (more or less) in the paper, making essential use of the metric completion (a fractal construction called there the *"Game of Thrones"* construction).

# The free $n$-form as 2d gaussian model (3)

The classical $n$-form theory on $M$ now *is* the classical 2d gaussian model on $\mathcal{Q} = \mathcal{D}^{int}_{n-1}(M)_{\partial\xi_0}$.

We have mathematically well-defined 1-forms and 0-forms on $\mathcal{Q}$ which are the pull backs of the $n$-forms and $(n-1)$-forms on $M$

$$j = \Pi_1^* F\,, \quad *j = \Pi_1^*(*F)\,, \quad \phi = \Pi_0^* A\,, \quad \phi^* = \Pi_0^* A^*$$

and we have mathematically well-defined classical equations of motion

$$j = d\phi\,, \qquad *j = d\phi^*$$

and vertex operators

$$V_{p,p^*}(\xi) = e^{ip\phi(\xi) + ip^*\phi^*(\xi)}$$

# Gauge symmetry

Each fiber of the bundle $\mathcal{Q}(M) \to \mathcal{PB}(M)$ carries a 2d gaussian model with a global $U(1) \times U(1)$ symmetry.

The collection of these $U(1) \times U(1)$ symmetries is a local gauge symmetry in the bundle of 2d theories over the base $\mathcal{PB}(M)$.

All gauge invariant expectation values can be calculated in any single fiber.

The huge local gauge symmetry on $\mathcal{PB}(M)$ reduces to the ordinary local space-time $U(1) \times U(1)$ gauge symmetry on $M$.

A major motivation of this project is the prospect of putting a 2d qft with *nonabelian* global symmetry on each of the fibers.

The collection of nonabelian global 2d symmetries is a nonabelian local symmetry over $\mathcal{PB}(M)$, which ought to/might give a nonabelian local gauge symmetry on space-time.

# Tomorrow

Define the intersection form on currents in $M$,

$$I_M(\xi_1, \xi_2) \neq 0 \quad \text{only if} \quad k_1 + k_2 = 2n$$

Pulled back to $\mathcal{Q}$,

$$\Pi^* I_M(\eta_1, \eta_2) \neq 0 \quad \text{only if} \quad j_1 + j_2 = 2$$

just as in a 2-manifold.

The currents in $\mathcal{Q}$ intersect like the currents in a 2-manifold, and there is a $*$-operator on 1-forms. Thus "quasi Riemann surface".

These are taken as the defining structures that make $\mathcal{Q}$ a "quasi Riemann surface".

Express quantization of the free $n$-form by the Schwinger-Dyson equation, which is written in terms of the $*$-operator and $I_M(\xi_1, \xi_2)$.

Then pull the S-D equation back to $\mathcal{Q}$ where it is identical to the S-D equation of the quantum 2d gaussian model, written in terms of the $*$-operator and $\Pi^* I_M(\eta_1, \eta_2)$.

So the quantum $n$-form on $M$ is the quantum 2d gaussian model on $\mathcal{Q}$.

On a Riemann surface, the S-D equation of the 2d gaussian model is just the Cauchy-Riemann equation

$$\bar{\partial}(1/z) = \pi i \delta^2(z)$$

that forms the basis for complex analysis on Riemann surfaces.

The formally identical S-D equation on $\mathcal{Q}$ should likewise form the basis for complex analysis on the quasi Riemann surfaces.

# Tomorrow (3)

We have the 2d gaussian model and the Cauchy-Riemann equation on the quasi Riemann surfaces $\mathcal{Q}$.

So now we have the essential tools to carry out all the constructions of 2d cft on the quasi Riemann surfaces.

Then we will have new cft's of extended objects in space-time, and can begin their study.

A cornucopeia of lovely problems.

# Side comment

The homotopy groups of $\mathcal{Q}$ are given by homology groups of $M$,

$$\pi_j(\mathcal{Q}) = H_{j+n-1}(M)$$

In particular

$$H_1(\mathcal{Q}) = \pi_1(\mathcal{Q}) = H_n(M)$$

If $H_1(\mathcal{Q})$ is non-trivial, then $\phi$ and $\phi^*$ will be multi-valued on $\mathcal{Q}$, just as in the 2d gaussian model on a Riemann surface $\Sigma$ with non-trivial $H_1(\Sigma)$.

On $M$, the obstruction to solving for $A$ globally is $H_n(M)$.

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# The intersection form on currents

The natural bilinear intersection form on currents in $M$ is

$$I_M(\xi_1, \xi_2) = \int_M \xi_1^{\mu_1 \cdots \mu_{k_1}}(x) \frac{\epsilon_{\mu_1 \cdots \mu_{k_1} \nu_1 \cdots \nu_{k_2}}(x)}{k_1! \, k_2!} \xi_2^{\nu_1 \cdots \nu_{k_2}}(x) \, d^{2n}x$$

- non-zero only if $k_1 + k_2 = 2n$
- well-defined on pairs of smooth currents
- well-defined on almost all pairs of integral currents
  (then giving the integer intersection number)

We want the S-D equations to look the same for all $n$. But the properties of $*$ and $I_M(\xi_1, \xi_2)$ depend on $n$. In particular,

$$*^2 = (-1)^n = \begin{cases} -1 & \text{for } n \text{ odd} \\ +1 & \text{for } n \text{ even} \end{cases}$$

To cope: complexify the currents (details suppressed) and modify $*$ and $I_M(\xi_1, \xi_2)$ slightly, so their properties become $n$-independent.

# Structures on complex currents in $M$

Define so that pulled back to $\mathcal{Q}$ their properties are $n$-independent:

$$J = \epsilon_n * \,, \qquad I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle = \epsilon_{n,k_2-n} I_M(\bar{\xi}_1, \xi_2)$$

$$\epsilon_n^2 = (-1)^{n-1}, \qquad \epsilon_{n,m} = (-1)^{nm+m(m+1)/2}\, \epsilon_n^{-1}$$

(For $n$ odd, $\epsilon_n$ is real. Then all currents can be taken real.)

Properties:

- $J^2 = -1$ (almost-complex structure)
- $I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle \neq 0$ only if $(k_1 - n + 1) + (k_2 - n + 1) = 2$
- $I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle$ is densely defined and nondegenerate
- $I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle = -\overline{I_M\langle\, \bar{\xi}_2, \xi_1 \,\rangle}$ (skew-hermitian)
- $I_M\langle\, \overline{\partial\xi_1}, \xi_2 \,\rangle + I_M\langle\, \bar{\xi}_1, \partial\xi_2 \,\rangle = 0$ (integration by parts)
- $I_M\langle\, \bar{\xi}_1, J\xi_2 \,\rangle$ on $n$-currents is hermitian and positive definite (i.e., on $k$-currents with $k - n + 1 = 1$)

Change basis to the chiral fields $F_\pm$, $A_\pm$

$$P_\pm = \frac{1}{2}\left(1 \pm i^{-1} J\right) , \qquad F_\pm = P_\pm F , \qquad dA_\pm = F_\pm$$

The euclidean adjoints are ($F$, $A$ are now complex)

$$F_\pm^\dagger = \overline{F_\mp} , \qquad A_\pm^\dagger = \overline{A_\mp} .$$

The Schwinger-Dyson equation is the same for all $n$,

$$\langle \int_{\bar{\xi}_0} A_\alpha^\dagger \int_{\xi_2} dF_\beta \rangle = -2\pi i \gamma_{\alpha\beta} I_M \langle \bar{\xi}_0, \xi_2 \rangle$$

$$\gamma_{++} = 1 , \quad \gamma_{--} = -1 , \quad \gamma_{+-} = \gamma_{-+} = 0$$

The lhs is the 2-point function $\langle A_\alpha^\dagger(x)\, dF_\beta(y) \rangle$ smeared against the $(0{+}n{-}1)$-current $\xi_0$ and the $(2{+}n{-}1)$-current $\xi_2$.
The rhs is a $\delta$-function in space-time smeared against the same currents.

# The S-D equation pulled-back to $\mathcal{Q}$

Pulled back to $\mathcal{Q} = \mathcal{D}^{int}_{n-1}(M)_{\mathbb{Z}\partial\xi_0}$ the S-D equation becomes formally identical to the S-D equation of the 2d gaussian model on a Riemann surface $\Sigma$

$$\langle \int_{\bar{\eta}_0} \phi^\dagger_\alpha \int_{\eta_2} dj_\beta \rangle = -2\pi i \gamma_{\alpha\beta} I_{\mathcal{Q}} \langle\, \bar{\eta}_0, \eta_2 \,\rangle$$

where $\eta_0$ is a 0-current and $\eta_2$ is a 2-current, and

$$I_{\mathcal{Q}} \langle\, \bar{\eta}_0, \eta_2 \,\rangle = \Pi^* I_M \langle\, \bar{\eta}_0, \eta_2 \,\rangle = I_M \langle\, \Pi_0 \eta_0, \Pi_2 \xi_2 \,\rangle$$

$I_{\mathcal{Q}} \langle\, \bar{\eta}_0, \eta_2 \,\rangle$ takes the place of $I_\Sigma \langle\, \bar{\eta}_0, \eta_2 \,\rangle$, the intersection form of the Riemann surface. (Think of $\eta_0$ a point, $\eta_2$ a disk.)

Unsmeared, the 2d S-D equation (for $\alpha = \beta = +$) is

$$\partial_{\bar{z}} \langle\, \phi^\dagger_+(w)\, j_+(z) \,\rangle = \pi i \delta^2(z - w)$$

which is the Cauchy-Riemann equation for $1/(z-w)$.

# Aside: the Dirac quantization condition $RR^* = 1$

The Dirac quantization condition $RR^* = 1$ is derived from the requirement that the correlation functions of the $V_{p,p^*}(\xi)$ should be single valued.

The derivation can be carried out on $\mathcal{Q}$ exactly as for the 2d gaussian model on a Riemann surface $\Sigma$.

Pick a point $\xi$ and a small disk $D$ in $\mathcal{Q}$, such that

$$I_{\mathcal{Q}}\langle\, [\xi], [D]\,\rangle = I_M\langle\, \xi, \Pi_2[D]\,\rangle \neq 0$$

The boundary $\partial D$ of the disk is a closed path in $\mathcal{Q}$.

The monodromy of $V_{p,p*}(\xi)V_{p',p*'}(\xi')$ as $\xi'$ moves around the closed path $\partial[D]$ can be calculated from the S-D equation on $\mathcal{Q}$.

The monodromy is trivial iff $R = R^*$.

In $M$, the $(n{-}1)$ current $\xi$ intersects the $(n{+}1)$-current $\Pi_2 D$. The $(n{-}1)$ current $\xi'$ sweeps out the $n$-boundary $\partial\Pi_2 D$.

# Definition of quasi Riemann surface

A quasi Riemann surface is

- an abelian group and a complete metric space $\mathcal{Q}$
- with a linear operator $J$ on 1-forms
- and a skew-hermitian form $I_{\mathcal{Q}}\langle\,\bar{\eta}_1, \eta_2\,\rangle$ on currents

such that

- $J^2 = -1$ (almost-complex structure)
- $I_{\mathcal{Q}}\langle\,\bar{\eta}_1, \eta_2\,\rangle \neq 0$ only if $j_1 + j_2 = 2$
- $I_{\mathcal{Q}}\langle\,\bar{\eta}_1, \eta_2\,\rangle$ is densely defined (but not non-degenerate)
- $I_{\mathcal{Q}}\langle\,\bar{\eta}_1, \eta_2\,\rangle = -\,\overline{I_{\mathcal{Q}}\langle\,\bar{\eta}_2, \eta_1\,\rangle}$ (skew-hermitian)
- $I_{\mathcal{Q}}\langle\,\overline{\partial\eta_1}, \eta_2\,\rangle + I_{\mathcal{Q}}\langle\,\bar{\eta}_1, \partial\eta_2\,\rangle = 0$ (integration by parts)
- $I_{\mathcal{Q}}\langle\,\bar{\eta}_1, J\eta_2\,\rangle$ on 1-currents is hermitian and non-negative

# Definition of quasi Riemann surface (2)

Divide out the $\mathcal{D}_j^{int}(\mathcal{Q})$ by the null spaces of $I_{\mathcal{Q}}\langle\, \bar{\eta}_1, \eta_2 \,\rangle$

$$\mathcal{Q}_j = \mathcal{D}_j^{int}(\mathcal{Q})/\mathcal{N}_j$$

so $I_{\mathcal{Q}}\langle\, \bar{\eta}_1, \eta_2 \,\rangle$ becomes non-degerate on $\bigoplus\limits_j \mathcal{Q}_j$.

(This is the physical equivalence relation on currents in $\mathcal{Q}$.)

For $\mathcal{Q} = \mathcal{D}_{n-1}^{int}(M)_{\mathbb{Z}\partial\xi_0}$ this is dividing by null spaces of $I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle$

$$\mathcal{Q}_{-1} = \mathbb{Z}\partial\xi_0 \subset \mathcal{D}_{n-2}^{int}(M)\,, \qquad \mathcal{Q}_0 = \mathcal{Q} = \mathcal{D}_{n-1}^{int}(M)_{\mathbb{Z}\partial\xi_0}$$

$$\mathcal{Q}_1 = \mathcal{D}_n^{int}(M)$$

$$\mathcal{Q}_2 = \mathcal{D}_{n+1}^{int}(M)/\mathcal{Q}_0^{\perp}\,, \qquad \mathcal{Q}_3 = \mathcal{D}_{n+2}^{int}(M)/\mathcal{Q}_{-1}^{\perp}$$

# Definition of quasi Riemann surface (3)

The $\mathcal{Q}_j$ form a chain complex

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{Q}_3 \arrow[r, "\partial"] \arrow[d, equal] & \mathcal{Q}_2 \arrow[r, "\partial"] & \mathcal{Q}_1 \arrow[r, "\partial"] & \mathcal{Q}_0 \arrow[r, "\partial"] \arrow[d, equal] & \mathcal{Q}_{-1} \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z} & & & \mathcal{Q} & \mathbb{Z} & \end{tikzcd}$$

endowed with almost-complex structure $J$ and skew-hermitian intersection form $I_{\mathcal{Q}}\langle\, \bar{\eta}_1, \eta_2 \,\rangle$

which is formally identical to the chain complex of integral currents $\mathcal{D}_j^{int} = \mathcal{D}_j^{int}(\Sigma)$ in a Riemann surface $\Sigma$ (augmented at both ends)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{D}_3^{int} \arrow[r, "\partial"] \arrow[d, equal] & \mathcal{D}_2^{int} \arrow[r, "\partial"] & \mathcal{D}_1^{int} \arrow[r, "\partial"] & \mathcal{D}_0^{int} \arrow[r, "\partial"] & \mathcal{D}_{-1}^{int} \arrow[r] \arrow[d, equal] & 0 \\ & \mathbb{Z} & & & & \mathbb{Z} & \end{tikzcd}$$

endowed with its almost-complex structure $J$ and skew-hermitian intersection form $I_{\Sigma}\langle\, \bar{\eta}_1, \eta_2 \,\rangle$.

A *"quasi holomorphic curve"* is a map $C\colon \Sigma \to \mathcal{Q}$ from a Riemann surface $\Sigma$ to $\mathcal{Q}$ that preserves the $J$-operators and the skew-hermitian forms.

The basic example, for $\mathcal{Q} = \mathcal{Q}(\Sigma) = \mathcal{D}_0^{int}(\Sigma)$,

$$C : z \in \Sigma \mapsto [z] \in \mathcal{D}_0^{int}(\Sigma)$$

Pulled back along $C$, the fields of the $n$-form theory on $\mathcal{Q}$ become *ordinary* conformal fields of the ordinary 2d gaussian model on $\Sigma$, because $C$ preserves the calculus of forms and the S-D equations.

# Local quasi holomorphic curves

A *local* qhc is a qhc with $\Sigma = \mathbb{D}$, the unit disk.

For each local qhc, we have the radial quantization, Virasoro algebras, and operator product expansion of the 2d cft on $\mathbb{D}$.

The set of all this local 2d cft data on the collection of all local qhc's is the local data of the cft on $\mathcal{Q}$.

Operator products of fields $V_1(\xi_1)$ and $V_2(\xi_2)$ on $\mathcal{Q}$ are seen locally as a collection of 2d operator products on each of the local qhc's.

Mathematically, the local properties of functions on $\mathcal{Q}$ will be expressed as the local properties as a function of one complex variable on each of the local qhc's.

# Some mathematical questions and speculations

1. Speculation: quasi Riemann surfaces are classified up to isomorphism by the Jacobian, i.e., the homology data in the middle dimension, $H_1(\mathcal{Q}) = H_n(M)$ as a lattice.

   For $M = S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$, the Jacobian is trivial, so the conjecture says that the quasi Riemann surfaces are isomorphic for all $n$,

   $$\mathcal{D}_0^{int}(S^2)_{\mathbb{Z}\partial\xi_0'} \;\simeq\; \mathcal{D}_{n-1}^{int}(S^{2n})_{\mathbb{Z}\partial\xi_0}$$

2. Speculation: every quasi Riemann surface $\mathcal{Q}$ is isomorphic to the integral currents $\mathcal{D}_0^{int}(\Sigma)$ in some 2-dimensional conformal space $\Sigma$.

   That is, a unique 2-dimensional space $\Sigma$ can be reconstructed from the $\mathcal{Q}_j$, the $J$-operator, and the skew-hermitian intersection form $I_{\mathcal{Q}}\langle\, \bar{\eta}_1, \eta_2 \,\rangle$.

   What are these spaces $\Sigma$ when the Jacobian is not that of a Riemann surface?

3. What are the automorphism groups of the quasi Riemann surfaces?

   If the above speculations are correct, then the automorphism group will depend only on the Jacobian. The automorphism groups will be very rich.

   For example, the automorphism group of the quasi Riemann surface associated to $S^2$, the quasi Riemann surface with trivial Jacobian, will include the conformal symmetry groups of all the $S^{2n}$.

4. How much of complex analysis can be done on quasi Riemann surfaces $\mathcal{Q}$, based on the Cauchy-Riemann equation written in terms of $J$ and $I_{\mathcal{Q}}\langle\,\bar{\eta}_1, \eta_2\,\rangle$ and/or the quasi holomorphic curves in $\mathcal{Q}$?

5. What can be said about the set of local quasi holomorphic curves in $\mathcal{Q}$? in particular, for the basic case $M = S^{2n}$?

# To do

1. Extend the 2d gaussian model from a Riemann surface $\Sigma$ to the quasi Riemann surface of integral 0-currents in $\Sigma$, $\mathcal{Q}(\Sigma) = \mathcal{D}_0^{int}(\Sigma)$.

   This requires renormalizing the vertex operators $V_{p,p^*}(\xi)$ for arbitrary 0-currents $\xi$.

   If the mathematical speculations are correct, then the divergence to be subtracted should be interpreted as the physical dimension for the 0-current $\xi$. The values of these physical dimensions should include all the dimensions ($n-1$ of the singular $(n-1)$ currents in $2n$ dimensions, for every $n$.

2. Try to do the same in the $n$-form theory, renormalizing the $V_{p,p^*}(\xi)$ for arbitrary integral $(n-1)$-currents $\xi$.

# To do (2)

3. Investigate the nonabelian $SU(2) \times SU(2)$ global symmetry in the 2d gaussian model on $\mathcal{Q}$ at the self-dual coupling $R = R^*$.

   Does the nonabelian local symmetry in the bundle $\mathcal{Q}(M) \to \mathcal{PB}(M)$ translate into a nonabelian local $SU(2) \times SU(2)$ gauge symmetry in space-time?

4. Try to extend other 2d cfts from a Riemann surface $\Sigma$ to the quasi Riemann surface of 0-currents in $\Sigma$, $\mathcal{Q}(\Sigma) = \mathcal{D}_0^{int}(\Sigma)$.

   Try to imitate on $\mathcal{Q}(\Sigma)$ the usual operations on 2d cfts, such as orbifolding, current algebra, chiral fermions, etc..

5. Try to imitate the usual operations on 2d cfts on a general quasi Riemann surface $\mathcal{Q}$, in particular on the $\mathcal{Q}(M)_{\mathbb{Z}\partial\xi_0}$.

6. What local cft's on $M$ are obtained on restriction to the small $(n{-}1)$-dimensional extended objects?