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Infrared properties of boundaries in one-dimensional quantum systems

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Abstract. We present some partial results on the general infrared behaviour of bulk critical 1D quantum systems with a boundary. We investigate whether the boundary entropy, $s(T)$, is always bounded below as the temperature T decreases towards 0, and whether the boundary always becomes critical in the infrared limit. We show that failure of these properties is equivalent to certain seemingly pathological behaviours far from the boundary. One of our approaches uses real time methods, in which locality at the boundary is expressed by analyticity in the frequency. As a preliminary, we use real time methods to prove again that the boundary beta function is the gradient of the boundary entropy, which implies that $s(T)$ decreases with T. The metric on the space of boundary couplings is interpreted as the renormalized susceptibility matrix of the boundary, made finite by a natural subtraction.

Keywords: rigorous results in statistical mechanics, renormalization group, surface effects (theory)

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Contents

1. Introduction

In this paper we study the infrared behaviour of bulk critical one-dimensional quantum systems with a boundary. These are 1D quantum systems whose bulk couplings are at a critical point, but whose boundary couplings are not necessarily critical. We would like to show that the boundary couplings are always driven to a renormalization group fixed point in the far infrared, which is to say that the boundary always becomes critical in the infrared limit. We would also like to show that the boundary entropy cannot decrease without limit, but must approach some lower bound as the temperature decreases towards zero. Alternatively, we would like to understand what kind of quantum boundary does *not* go to an IR fixed point, or *does* release an unlimited amount of entropy as its temperature goes to zero. We record here some partial results which might be useful as steps towards these goals.

The boundary entropy, s, is the difference between the total entropy and the bulk entropy (which is proportional to the length of the system). For critical boundaries, the number $g=\exp(s)$ is the universal non-integer ground state degeneracy of Affleck and Ludwig [1]. In [3], we proved a gradient formula

$$\frac{\partial s}{\partial\lambda^a}=-g_{ab}(\lambda)\beta^b(\lambda) \tag{1}$$

which expresses the boundary beta function, β^b, as the gradient of the boundary entropy, s, with respect to a certain metric, g_{ab}, on the space of all the marginal and relevant boundary couplings. The λ^a are the boundary coupling constants. The boundary entropy depends on the temperature and the boundary couplings, and satisfies the renormalization group equation

$$\left(T\frac{\partial}{\partial T}+\beta^a\frac{\partial}{\partial\lambda^a}\right)s=0, \tag{2}$$

so the boundary gradient formula implies that

$$T\frac{\partial s}{\partial T}=\beta^a g_{ab}\beta^b\geq 0. \tag{3}$$

Thus $s(T)$ always decreases with decreasing temperature, which is to say that the boundary entropy always decreases under the renormalization group. The boundary is critical, $\beta^a(\lambda)=0$, if and only if the boundary entropy is stationary in the temperature, $\mathrm{d}s/\mathrm{d}T=0$. The boundary entropy can decrease below zero because the third law of thermodynamics does not apply. The boundary is not an isolated system.

We would like to understand the properties of the boundary in the far infrared. For bulk 1D quantum systems, without boundary, the c-theorem [7] gives considerable control over the infrared behaviour. The c-theorem states that a certain function of the bulk couplings decreases under the renormalization group, is stationary if and only if the bulk beta function vanishes, and cannot become negative. This is almost enough to show that the bulk system must flow to a fixed point in the infrared. We point out below an additional assumption that is needed.

The generic bulk system has a mass gap, so it flows in the infrared to the trivial $c=0$ fixed point, where no excitations remain. There does not seem to be an analogously trivial boundary system. A boundary that flowed to $s=-\infty$, $g=0$ might provide a candidate, but no such system is known. In every known example, the infrared limit is a non-trivial boundary fixed point and the boundary entropy decreases to a finite lower limit. Non-trivial boundary excitations always remain. It can be conjectured that the boundary entropy is necessarily bounded below throughout a RG flow, and that the flow necessarily ends at an IR fixed point, unless some pathologies develop. We would like to understand what technical assumptions are needed to prove these conjectures, and what physical principles they express.

The boundary gradient formula succeeds in excluding some exotic forms of renormalization group behaviour. For example, limit cycles within the space of boundary couplings are impossible. But the gradient formula by itself does not guarantee that the system flows to an infrared fixed point. The boundary entropy might decrease without bound, with $\beta^a(\lambda)$ never approaching zero. This possibility could be excluded if we could show that the boundary entropy is bounded below (for a given bulk critical system). We are at least able to show, under certain assumptions, that $\beta^a g_{ab}\beta^b\to 0$ in the infrared limit (see section 6). This is analogous to what the c-theorem provides in the bulk. It does not establish that β^a vanishes in the infrared limit, just as the analogous bulk result does not, but this is a step in the right direction.

A lower bound on the boundary entropy would also be of interest because it would imply that only a bounded amount of information can be added to a given boundary or junction within a near-critical quantum circuit. Such circuits have been argued to be the ideal physical systems for asymptotically large-scale quantum computers [4]. A lower bound on the boundary entropy would be a very general constraint on the design of such quantum computers.

There are a number of examples of a lower bound on $g = \exp(s)$ for boundary conformal field theories corresponding to a given bulk conformal field theory. For the compact $U(1)$ Gaussian model with target radius R, normalized so that $R = 1$ is the self-dual radius, the lowest value of s corresponds to the Dirichlet boundary condition, $s_{\mathrm{D}} = -\frac{1}{4}\ln 2 - \frac{1}{2}\ln R$, when $R \geq 1$, and to the Neumann boundary condition, $s_N = -\frac{1}{4}\ln 2 + \frac{1}{2}\ln R$, when $R \leq 1$, so the lower bound on s is $-\frac{1}{4}\ln 2 - \frac{1}{2}|\ln R|$. Clearly, there is no universal lower bound, independent of the bulk conformal field theory.

Another set of examples is provided by the conformal boundary conditions given by the Cardy boundary states in rational conformal field theories [6]. Each Cardy boundary state is labelled by a primary field i. We point out in the appendix that the Cardy state with the smallest value of s is the one associated with the identity operator, $i = 0$, so the lower bound on s is $s_0 = \frac{1}{2}\ln S_{00}$ where S_{00} is the corresponding entry of the modular S-matrix. In the case of the unitary $c < 1$ conformal field theories, the Cardy states are all the possible conformal boundary conditions. For the unitary minimal models with central charge

$$c_m = 1 - \frac{6}{m(m+1)}, \qquad m = 2, 3, \ldots$$

the lower bound is

$$s_0(m) = \frac{1}{4}\ln\left[\frac{8}{m(m+1)}\sin^2\left(\frac{\pi}{m}\right)\sin^2\left(\frac{\pi}{m+1}\right)\right].$$

In these examples, one can observe the crucial role of locality in putting a lower bound on s. It is the imposition of the Cardy constraint, which is a form of the locality condition, that ensures a non-zero overlap $g = \langle B|0\rangle$ between the boundary state and the conformal vacuum.

In this paper, we start by arguing that any critical boundary system must have $g > 0$, or else the system would not have a sensible thermodynamic limit. We then argue that, for non-critical boundaries, the boundary contribution, $\theta(\tau)$, to the trace of the energy–momentum tensor goes to a multiple of the identity operator in the far infrared. We work directly at $T = 0$. Specifically, we show that its connected two-point function in Euclidean time satisfies

$$\lim_{\tau\to\infty} \tau^2 \langle\theta(\tau)\,\theta(0)\rangle_{\mathrm{c}} = 0. \tag{4}$$

We need to assume that, far from the boundary, the bulk conformal invariance is restored in a strong sense. The canonical scaling dimension of $\theta(\tau)$ is 1, so equation (4) comes close to implying that $\theta(\tau)$ vanishes up to a multiple of the identity operator, which would imply that the infrared limit is scale invariant. To finish the argument, we need that the correlation functions of the bulk operators satisfy a cluster decomposition condition in the infrared limit. This is essentially the assumption that the infrared limit is a well-defined boundary quantum field theory, in which case the vanishing of the two-point function implies the vanishing of the operator. We do not know whether our assumption is provable from general principles. If this gap can be filled, then the infrared limit at $T = 0$ is a boundary quantum field theory with $\theta(\tau) = \langle\theta\rangle\,\mathbf{1}$, which is a boundary

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conformal field theory. Given the previous argument that any boundary conformal field theory has $s > -\infty$, the boundary entropy of the original system would be bounded below.

An analogous gap exists in the argument that the infrared limit in the bulk is always a fixed point. An assumption is also needed that the infrared limit is a well-defined quantum field theory, so that the vanishing of the two-point function of the trace of the energy–momentum tensor implies that the operator itself vanishes. In the boundary case, the bulk operator algebra does not change under the renormalization group, so the situation might be better than in the bulk case. This leaves a hope that our results can be strengthened.

Our second approach is to use real time methods at $T > 0$. As a preliminary step, we re-prove the boundary gradient formula using real time methods, based on the spectral analysis of the flow of entropy through the boundary [5]. In this version of the proof, the metric g_{ab} is given a physical interpretation. It is the renormalized boundary susceptibility matrix, made finite by a natural subtraction. It can be measured experimentally. We try to use the real time formalism to show that $\mathrm{d}s/\mathrm{d}T = \beta^a g_{ab} \beta^b / T$ is integrable with respect to T at $T = 0$. This would imply a lower bound on s. We only succeed in showing that $T\,\mathrm{d}s/\mathrm{d}T \to 0$ as $T \to 0$, which implies that $\beta^a g_{ab} \beta^b \to 0$. The condition of integrability at $T = 0$ is reformulated as an estimate on the low temperature behaviour of a certain spectral function, an estimate that we do not know how to prove.

2. Notation and basic facts

We will be using both real and Euclidean time descriptions of a one-dimensional quantum system. The space–time coordinates are (x, t), $x \geq 0$. The boundary is at $x = 0$. The Euclidean time is $\tau = \mathrm{i}t$. The space–time metric is

$$(\mathrm{d}s)^2 = -v^2(\mathrm{d}t)^2 + (\mathrm{d}x)^2 = v^2(\mathrm{d}\tau)^2 + (\mathrm{d}x)^2$$

where v is the velocity of 'light'. The system is in equilibrium at temperature T. The imaginary time correlation functions are periodic in Euclidean time, with period $\beta = 1/T$ (in units with $\hbar = k = 1$). The normalized equilibrium expectation values are denoted by $\langle \mathcal{O} \rangle_{\mathrm{eq}}$. The connected two-point expectation values are $\langle \mathcal{O}_1\, \mathcal{O}_2 \rangle_{\mathrm{c}} = \langle \mathcal{O}_1\, \mathcal{O}_2 \rangle_{\mathrm{eq}} - \langle \mathcal{O}_1 \rangle_{\mathrm{eq}} \langle \mathcal{O}_2 \rangle_{\mathrm{eq}}$. The energy–momentum tensor is $T^\mu_\nu(x, t)$. Conservation of energy–momentum in the bulk is expressed by

$$\partial_\mu T^\mu_\nu(x, t) = 0 \qquad x > 0. \tag{5}$$

The Hamiltonian is

$$H = -\theta(t) + \int_0^\infty \mathrm{d}x\, T^t_t(x, t) \tag{6}$$

where $-\theta(t)$ is the boundary energy operator. Energy conservation at the boundary is[1]

$$\partial_t \theta(t) = T^x_t(0, t). \tag{7}$$

[1] In the present paper our conventions differ from the ones in [3] in that the energy–momentum components have canonical dimensions, instead of being dimensionless, as in [3]. As a result, extra factors of the RG scale μ are present in various equations in [3].

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The energy density $T^t_t(x,t)$ is the only component of the energy–momentum tensor that has a boundary contribution. See [3] for a more complete discussion of the bulk + boundary energy–momentum tensor.

Bulk criticality is equivalent to local scale invariance in the bulk:

$$\Theta(x,t) = T^\mu_\mu(x,t) = T^x_x(x,t) + T^t_t(x,t) = 0 \qquad x > 0. \tag{8}$$

The trace of the energy–momentum tensor is concentrated at the boundary,

$$\Theta(x,t) = \delta(x)\theta(t), \tag{9}$$

and can be expanded in the boundary fields:

$$\theta(t) = \beta^a \phi_a(t) \tag{10}$$

where the boundary operators $\phi_a(t)$ are the relevant and marginal fields localized at the boundary. The coefficients β^a comprise the boundary beta function. The operators $\phi_a(t)$ have ultraviolet scaling dimensions all ≤ 1. The boundary coupling constants, λ^a, are related to the boundary fields, $\phi_a(t)$, by

$$\frac{\partial Z}{\partial \lambda^a} = \frac{\partial z}{\partial \lambda^a} = \beta \langle \phi_a(0)\rangle_{\mathrm{eq}} \tag{11}$$

where Z is the full partition function and z is the boundary partition function. The definition of z starts with a system of finite length, L. An arbitrary boundary condition is imposed at $x = L$. In the thermodynamic limit $L \to \infty$, the full partition function, Z_L, factorizes into a bulk part and a boundary part:

$$\mathrm{e}^{-\pi c L/6\beta} Z_L \to z z'$$

where c is the bulk conformal central charge and the constants z and z' are the boundary partition functions of the boundaries at $x = 0$ and $x = L$ respectively. Only the product zz' is determined. Unitarity of the quantum system implies that all the products zz' are real and positive, for all pairs of boundary conditions. We can take the boundary condition at $x = L$ to be the same as the boundary condition at $x = 0$ (strictly speaking, the CPT transform of the boundary condition at $x = 0$). Then $\mathrm{e}^{-\pi c L/6\beta} Z_L \to |z|^2$. Now we can determine z as the positive real square root of $|z|^2$. This is consistent, because all the products zz' are positive real numbers. We construct the system on the infinite half-cylinder with a single boundary at $x = 0$ by taking the limit $L \to \infty$, dividing by z' to eliminate dependence on the boundary condition at $x = L$. In terms of the bulk conformal field theory on the half-cylinder, where the spatial coordinate is $v\tau$ and the Euclidean time is x/v, the boundary condition at $x = 0$ is represented by a boundary state $\langle B|$, while the 'boundary condition' at $x = L$ is represented by the bulk ground state $|0\rangle$, since all the excited states at $x = L$ are suppressed exponentially in L. The boundary partition function is the overlap $z = \langle B|0\rangle$. The logarithm of the full partition function then takes the form

$$\ln Z_L = \frac{c\pi}{6\beta} L + \ln z \tag{12}$$

where $c\pi/6\beta$ is the universal ground state energy density of the bulk conformal field theory.

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The total entropy of the system is $S_L = (1 - \beta\partial/\partial\beta)Z_L$. Removing the bulk contribution leaves the boundary entropy

$$s = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln z. \tag{13}$$

The boundary entropy is a function of $\mu\beta$, where μ is the renormalization scale. It satisfies the renormalization group equation

$$\left(-\mu\frac{\partial}{\partial\mu} + \beta^a\frac{\partial}{\partial\lambda^a}\right)s = \left(-\beta\frac{\partial}{\partial\beta} + \beta^a\frac{\partial}{\partial\lambda^a}\right)s = 0. \tag{14}$$

For thermodynamic quantities, the infrared limit $\mu \to \infty$ is equivalent to the zero-temperature limit $T \to 0$. In this paper, we will avoid writing μ and $\mu \to \infty$. Instead, when we study thermodynamic quantities, we will take $T \to 0$, and use the second form of the renormalization group equation for s. When we study the quantum field theory at $T = 0$, we will take the IR limit by scaling all times and distances to infinity in the correlation functions.

The boundary beta function vanishes at a fixed point, so s is then a number, independent of temperature: $s = \ln z = \ln g$, where g is the universal non-integer ground state degeneracy of Affleck and Ludwig [1]. This is the 'ground state' degeneracy because, being constant in T, it can be evaluated at $T = 0$. For any finite L, the energy spectrum is discrete, so the ground state degeneracy is then an integer. The spectrum becomes continuous in the limit $L \to \infty$, so the numerical factor $z = g$ can be an arbitrary non-negative number. In particular, it is possible to have $g < 1$, $s < 0$.

Affleck and Ludwig conjectured that the value of g is larger at the ultraviolet fixed point of a renormalization group trajectory than at the infrared fixed point [1, 2]. This *g-theorem* was proved in [3] by proving the boundary gradient formula, equation (1). The boundary gradient formula implies that the boundary entropy decreases with decreasing temperature, $\mathrm{d}s/\mathrm{d}T > 0$, so the boundary entropy decreases along the renormalization group trajectory, so the value of $s = \ln g$ at the ultraviolet fixed point, at $T = \infty$, is greater than the value at the infrared fixed point, at $T = 0$. Ordinary entropy in statistical mechanics always decreases with temperature, but this is not obvious for the boundary entropy. The total entropy S_L of the system of length L does go down with temperature, trivially, but the large bulk contribution, $c\pi L/3\beta$, also decreases with temperature, so it is not obvious that the difference, the boundary entropy, decreases with temperature.

The metric in the gradient formula is

$$g_{ab} = \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' \langle\phi_a(\tau)\phi_b(\tau')\rangle_c \left[1 - \cos\left(\frac{2\pi(\tau - \tau')}{\beta}\right)\right] \tag{15}$$

so

$$\frac{\mathrm{d}s}{\mathrm{d}T} = \frac{1}{T}\int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau' \,\langle\theta(\tau)\theta(\tau')\rangle_c \left[1 - \cos\left(\frac{2\pi(\tau - \tau')}{\beta}\right)\right]. \tag{16}$$

Canonical ultraviolet behaviour ensures that any non-universal contact terms in the two-point function have dimension at most 2. The factor $1 - \cos(2\pi(\tau - \tau')/\beta)$ vanishes to second order at $\tau = \tau'$, so no contact terms contribute to the metric. The metric is thus finite and universal, assuming canonical ultraviolet behaviour. However, it is difficult to

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see a physical interpretation of the metric when it is written in this form, as an integral of a two-point function over Euclidean time.

Given bulk conformal invariance, the symmetric energy–momentum tensor has only two independent components:

$$\begin{aligned} T_t^x(x,t) &= -v^2 T_x^t(x,t) = T_R(x,t) - T_L(x,t), \\ vT_t^t(x,t) &= -vT_x^x(x,t) = T_R(x,t) + T_L(x,t). \end{aligned} \tag{17}$$

The bulk conservation law implies that $T_R(x,t)$ and $T_L(x,t)$ are chiral currents:

$$T_R(x,t) = T_R(x - vt), \qquad T_L(x,t) = T_L(x + vt). \tag{18}$$

They are related to the Virasoro operators in the 'closed string' channel:

$$\begin{aligned} T_R(z) &= T_{zz}(z) = -\frac{v^2}{2\pi} T(z) = -\frac{2\pi}{\beta^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi n z / v\beta} L_n, \\ T_L(\bar{z}) &= T_{\bar{z}\bar{z}}(\bar{z}) = -\frac{v^2}{2\pi} \bar{T}(\bar{z}) = -\frac{2\pi}{\beta^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi n \bar{z} / v\beta} \bar{L}_n \end{aligned} \tag{19}$$

where $z = x + \mathrm{i}v\tau = x - vt$. The coefficients are fixed by calculating the Hamiltonian in the 'closed string' channel, where $v\tau$ is the spatial coordinate and x/v the Euclidean time:

$$H_{\mathrm{closed}} = \frac{2\pi}{\beta}(L_0 + \bar{L}_0) = \int_0^\beta \mathrm{d}\tau \; vT_x^x.$$

On the semi-infinite cylinder, the boundary condition at $x = \infty$ is the bulk ground state, which satisfies $L_n|0\rangle = \bar{L}_n|0\rangle = 0$, $n \geq -1$. This implies that the bulk energy–momentum tensor, within correlation functions, decreases at infinity as

$$T_\nu^\mu(x,\tau) \sim \mathrm{e}^{-4\pi x/\beta}, \qquad x \to \infty. \tag{20}$$

Energy conservation at the boundary becomes

$$\partial_t \theta(t) = T_R(-vt) - T_L(vt). \tag{21}$$

Therefore

$$\begin{aligned} \theta(t) &= \int_{-\infty}^{t} \mathrm{d}t' \; T_R(-vt') - \int_{-\infty}^{t} \mathrm{d}t' \; T_L(vt') \\ &= \int_t^\infty \mathrm{d}t' \; T_L(vt') - \int_t^\infty \mathrm{d}t' \; T_R(-vt'). \end{aligned} \tag{22}$$

From (6), (17), (22) we obtain

$$H = \int_{-\infty}^{\infty} \mathrm{d}t \; T_R(vt) = \int_{-\infty}^{\infty} \mathrm{d}t \; T_L(vt). \tag{23}$$

3. Boundary entropy at a fixed point and locality

We argue that all critical boundaries have $g > 0$. Suppose otherwise. Then there would be a conformal boundary condition given by a boundary state $|B\rangle$ such that $g = \langle 0|B\rangle = 0$. In this case we should re-examine the $L \to \infty$ thermodynamic limit. The boundary state $|B\rangle$ is put at $x = 0$. At the other boundary, at $x = L$, we put a boundary condition $\langle B'|$. We choose $\langle B'|$ with the property that $\langle B'|0\rangle > 0$ so as to ensure that a conformal vacuum remains when we take the limit $L \to \infty$. The one-point functions of bulk operators are then defined as

$$\langle \phi(x,t)\rangle = \lim_{L\to\infty} \frac{\langle B'|\mathrm{e}^{-LH_{\mathrm{closed}}}\phi(x,t)|B\rangle}{\langle B'|\mathrm{e}^{-LH_{\mathrm{closed}}}|B\rangle}. \tag{24}$$

The numerator in this fraction goes as $\mathrm{e}^{L\pi c/6}$ for large L, while the denominator goes as $\mathrm{e}^{L(\pi c/6-\Delta_1)}$ where Δ_1 is the lowest eigenvalue occurring in the action of H_{closed} on $|B\rangle$. If $\langle 0|B\rangle = 0$, then $\Delta_1 > 0$ and the above limit is infinite, which means that there is no sensible thermodynamic limit. Alternatively, we could try defining a thermodynamic limit by putting the boundary state $|B\rangle$ on both ends of the cylinder. In the limit $L \to \infty$, we would obtain finite correlation functions on the infinite half-cylinder, but these correlation functions would generically grow exponentially with separation and thus violate cluster decomposition in the x-direction. So $g > 0$ for any sensible boundary conformal field theory.

4. At $T = 0$, $\lim_{\tau\to\infty} \tau^2\langle\theta(\tau)\theta(0)\rangle_c = 0$

Next, we try to argue that every boundary system flows to an infrared fixed point: a scale invariant, conformally invariant boundary field theory. Then, by the argument above, the boundary entropy would necessarily be bounded below, because the infrared fixed point would have $g > 0$.

We work directly at $T = 0$. The Euclidean space–time is the half-plane, $x \geq 0$, $-\infty < \tau < \infty$. We argue that

$$\lim_{|\tau-\tau'|\to\infty} |\tau-\tau'|^2\langle\theta(\tau)\theta(\tau')\rangle_c = 0. \tag{25}$$

Here $\langle\theta(\tau)\theta(\tau')\rangle_c$ is the zero-temperature connected correlator evaluated on the boundary of the infinite half-plane. The factor $|\tau-\tau'|^2$ accounts for the canonical scaling dimension of $\theta(\tau)$.

If we can assume that the infrared limit is a boundary quantum field theory, then we can conclude from equation (25) that $\theta(t)$ is a multiple of the identity in that limiting theory, so the infrared limit is a conformally invariant boundary quantum field theory, a fixed point of the renormalization group.

The last assumption, that the infrared limit is a quantum field theory, is also implicitly present when the c-theorem is used to show that every bulk quantum field theory goes to a fixed point (perhaps trivial) in the infrared. The c-theorem [7] implies that the trace, $\Theta = T^{\mu}_{\mu}$, of the bulk energy–momentum tensor has a vanishing connected two-point function in the infrared limit. This in turn implies that all correlation functions of the limiting theory are conformally invariant. The implicit assumption is that those correlation functions exist in the infrared limit. In the boundary case the situation might be more favourable, because the bulk operator algebra stays fixed (is not flowing). This leaves a hope that our results can be strengthened.

Our argument is based on the principle that the system should become conformally invariant far from the boundary. Consider the quantization in which τ is the spatial coordinate and x is the Euclidean time (call this the x-quantization). Space is now the entire real line, $-\infty < \tau < \infty$. The boundary condition is represented by a state $|\mathcal{B}\rangle$ inserted at $x = 0$. The correlation functions are expectation values of x-ordered products of operators:

$$\langle \phi_1(\tau_1, x_1) \dots \phi_n(\tau_n, x_n) \rangle_{\mathcal{B}} = \langle 0|\phi_1(\tau_1, x_1) \dots \phi_n(\tau_n, x_n)|\mathcal{B}\rangle \tag{26}$$

where $\langle 0|$ is here the vacuum state, and $x_1 \geq x_2 \geq \dots \geq x_n$. The correlation functions are normalized, $\langle 0|\mathcal{B}\rangle = 1$.

Suppose Q is the generator of a symmetry of the bulk system, specifically a global conformal symmetry. Then $\langle 0|Q = 0$. It seems reasonable to suppose that

$$\langle 0|Q\phi_1(\tau_1, x_1) \dots \phi_n(\tau_n, x_n)|\mathcal{B}\rangle = 0. \tag{27}$$

For the bulk global conformal symmetry group, $SL(2, \mathbb{C})$, we can take Q to be any of the six generators

$$\begin{aligned} Q_n &= \int_{-\infty}^{\infty} \mathrm{d}\tau (x + \mathrm{i}v\tau)^n T_R(x + \mathrm{i}v\tau), \\ \bar{Q}_n &= \int_{-\infty}^{\infty} \mathrm{d}\tau\, (x - \mathrm{i}v\tau)^n \bar{T}_L(x - \mathrm{i}v\tau) \qquad n = 0, 1, 2. \end{aligned} \tag{28}$$

We should note that there is a subtlety in the above reasoning. A conserved charge Q is defined as an integral

$$Q = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, j^x(\tau, x) = \lim_{R\to\infty} \int_{|\tau|<R} \mathrm{d}\tau\, j^x(\tau, x) \tag{29}$$

where $j^x(\tau, x)$ is the x-component of the corresponding current. Since we have very little knowledge of the properties of the state $|\mathcal{B}\rangle$ in general, we can worry that the limit $R \to \infty$ taken in a correlator

$$\lim_{R\to\infty} \langle 0| \int_{|\tau|<R} \mathrm{d}\tau j^x(\tau, x)\phi_1(\tau_1, x_1) \dots \phi_n(\tau_n, x_n)|\mathcal{B}\rangle \tag{30}$$

might not converge to zero. The problem with this limit could be due to a high density of low energy states present in $|\mathcal{B}\rangle$. If for some reason the above limit does not converge to zero this would mean that the asymptotic symmetry Q is spontaneously broken by the boundary condition $|\mathcal{B}\rangle$. We will assume that this does not happen or, in other words, the charge Q exists and is an asymptotic symmetry in the theory on a half-plane with the given boundary condition $|\mathcal{B}\rangle$. The condition $\langle 0|Q = 0$, understood in the above sense, implies that correlation functions are asymptotically conformally invariant. That is, correlation functions containing a commutator $[Q, \phi(\tau, x)]$ asymptotically vanish for $x \to \infty$. But the condition $\langle 0|Q = 0$ is stronger.

At temperature $T > 0$, the Euclidean time is compact, so there is no subtlety in expressing the bulk conformal invariance. In appendix A we use the bulk conformal invariance at $T > 0$, then take the $T \to 0$ limit, and, assuming that dispersion relations

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behave in a continuous fashion in this limit, we reproduce all the consequences of the $\langle 0|Q = 0$ assumption that we are making here. It cannot be considered as a derivation of the $\langle 0|Q = 0$ condition, though, because the assumption of continuity at $T = 0$ is essentially as strong as the $\langle 0|Q = 0$ condition itself.

With this assumption, we can write, for any $x > 0$,

$$0 = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\tau\, (x + \mathrm{i}v\tau)^n \langle T_R(x + \mathrm{i}v\tau)\theta(0)\rangle_c & n = 0, 1, 2 \\ \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\tau\, (x - \mathrm{i}v\tau)^n \langle T_L(x - \mathrm{i}v\tau)\theta(0)\rangle_c & n = 0, 1, 2 \end{cases} \tag{31}$$

or, equivalently,

$$0 = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\tau\, \tau^n \langle T_R(x + \mathrm{i}v\tau)\theta(0)\rangle_c & n = 0, 1, 2 \\ \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}\tau\, \tau^n \langle T_L(x - \mathrm{i}v\tau)\theta(0)\rangle_c & n = 0, 1, 2. \end{cases} \tag{32}$$

Now we consider the spectral representations for the two-point correlation functions. In Euclidean time we have

$$\begin{aligned} \langle T_R(x + \mathrm{i}v\tau)\theta(0)\rangle_c &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\omega\, [\theta(\tau)\theta(\omega) - \theta(-\tau)\theta(-\omega)] \mathrm{e}^{-\omega(\tau - \mathrm{i}x/v)} A_{\theta R}(\omega) \\ \langle T_L(x - \mathrm{i}v\tau)\theta(0)\rangle_c &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\omega\, [\theta(-\tau)\theta(\omega) - \theta(\tau)\theta(-\omega)] \mathrm{e}^{\omega(\tau + \mathrm{i}x/v)} A_{\theta L}(\omega) \end{aligned} \tag{33}$$

$$\langle \theta(\tau)\theta(0)\rangle_c = \frac{1}{2\pi} \int_{0}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-\omega|\tau|} A_{\theta\theta}(\omega). \tag{34}$$

The boundary conservation equation (7), written as

$$\partial_\tau \theta(\tau) = -\mathrm{i}[T_R(\mathrm{i}v\tau) - T_L(-\mathrm{i}v\tau)], \tag{35}$$

implies[2]

$$A_{\theta\theta}(\omega) = \omega^{-1}(A_{\theta R}(\omega) + A_{\theta L}(-\omega)) = \omega^{-1}(A_{\theta R}(-\omega) + A_{\theta L}(\omega)). \tag{36}$$

We note, though we do not use it here, that $T_R(x - vt)$ and $T_L(x + vt)$ are self-adjoint operators, so

$$\overline{A_{\theta R}(\omega)} = A_{\theta R}(-\omega), \qquad \overline{A_{\theta L}(\omega)} = A_{\theta L}(-\omega). \tag{37}$$

and, by reflection positivity, $A_{\theta\theta}(\omega) \geq 0$.

The spectral functions $A_{\theta R}(\omega)$, $A_{\theta L}(\omega)$ are related to the commutators as

$$\begin{aligned} A_{\theta R}(\omega) &= \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle \mathrm{i}[T_R(0, t), \theta(0)]\rangle = \int_{0}^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle \mathrm{i}[T_R(0, t), \theta(0)]\rangle, \\ A_{\theta L}(\omega) &= \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle -\mathrm{i}[T_L(0, t), \theta(0)]\rangle = \int_{-\infty}^{0} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle -\mathrm{i}[T_L(0, t), \theta(0)]\rangle \end{aligned} \tag{38}$$

[2] In deriving this equation from the boundary conservation equation one uses the fact that the correlator $\langle \theta(\tau)\theta(\tau')\rangle_c$ vanishes for large separation and hence there cannot be a term proportional to $\delta(\omega)$ in the spectral function.

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where the final forms of the equations are consequences of the chirality of the energy–momentum currents, equation (18), and causality (the vanishing of equal-time commutators at non-zero separation). It follows from the final forms of equations (38) that the spectral functions $A_{\theta R}(\omega)$ and $A_{\theta L}(\omega)$ are analytic in the complex upper half-plane. Again, we note but do not use here that energy conservation at the boundary combined with the bulk equal-time commutation relations of the chiral energy–momentum currents now imply $A_{\theta R}(\omega) = A_{\theta L}(\omega)$, so $A_{\theta\theta}(\omega) = (2/\omega)\mathrm{Re}\, A_{\theta R}(\omega)$.

The conformal invariance of the bulk vacuum at large x, expressed by equations (32), is equivalent to

$$0 = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega x/v} \omega^{-n-1} A_{\theta R}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega x/v} \omega^{-n-1} A_{\theta L}(\omega), \qquad n = 0, 1, 2. \tag{39}$$

It follows from (37) and (39) that

$$0 = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \omega^{-3} [\sin(\omega x/v)\, \mathrm{Re}\, A_{\theta R}(\omega) + \cos(\omega x/v)\, \mathrm{Im}\, A_{\theta R}(\omega)]. \tag{40}$$

This implies that the functions $\mathrm{Re}\, A_{\theta R}(\omega)/\omega^2$ and $\mathrm{Im}\, A_{\theta R}(\omega)/\omega^3$ are integrable at $\omega = 0$. This implies in particular that

$$\lim_{\omega\to 0} \frac{\mathrm{Im}\, A_{\theta R}(\omega)}{\omega^2} = 0. \tag{41}$$

Also, taking $x \to 0$ in (40), we obtain a sum rule

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\mathrm{Im}\, A_{\theta R}(\omega)}{\omega^3} = 0. \tag{42}$$

The just derived integrability properties of $\mathrm{Im}\, A_{\theta R}(\omega)$ and $\mathrm{Re}\, A_{\theta R}(\omega)$ at $\omega = 0$, and canonical UV behaviour, and analyticity in the upper half-plane allow one to write the following subtracted dispersion relations:

$$\begin{aligned} \frac{\mathrm{Re}\, A_{\theta R}(\omega)}{\omega^2} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\eta\, \frac{\mathrm{Im}\, A_{\theta R}(\eta)}{\eta^2} \mathcal{P}\left(\frac{1}{\eta - \omega}\right), \\ \frac{\mathrm{Im}\, A_{\theta R}(\omega)}{\omega^2} &= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\eta\, \frac{\mathrm{Re}\, A_{\theta R}(\eta)}{\eta^2} \mathcal{P}\left(\frac{1}{\eta - \omega}\right) \end{aligned} \tag{43}$$

for $\omega \neq 0$. The integrability of $\mathrm{Im} A_{\theta R}(\omega)/\omega^3$ at $\omega = 0$ allows us to take the limit $\omega \to 0$ of the first dispersion relation in (43) in a straightforward way and we obtain

$$\lim_{\omega\to 0} \frac{\mathrm{Re}\, A_{\theta R}(\omega)}{\omega^2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\eta\, \frac{\mathrm{Im}\, A_{\theta R}(\eta)}{\eta^3}.$$

This equation together with the sum rule (42) and equation (41) imply that

$$\lim_{\omega\to 0} \frac{A_{\theta R}(\omega)}{\omega^2} = 0. \tag{44}$$

By the same argument,

$$\lim_{\omega\to 0} \frac{A_{\theta L}(\omega)}{\omega^2} = 0. \tag{45}$$

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Therefore by (36)

$$\lim_{\omega\to 0} \frac{A_{\theta\theta}(\omega)}{\omega} = 0 \tag{46}$$

which in turn implies (25), which was to be shown.

Noting that $\theta(\tau)$ has a canonical scaling dimension 1, we infer that in the infrared limit $\mu \to \infty$ the two-point function at hand goes to zero:

$$\lim_{\mu\to\infty} \langle \theta(\tau)\theta(0)\rangle_c = 0. \tag{47}$$

In a quantum field theory, a local field with vanishing two-point function annihilates the ground state, and therefore has vanishing correlation functions with all other fields. Thus, if we can assume that we obtain a local boundary quantum field theory in the infrared limit $\mu \to \infty$, and if we can assume that

$$\langle 0|Q\phi_1(\tau_1, x_1)\ldots\phi_n(\tau_n, x_n)|\mathcal{B}\rangle = 0$$

for all the bulk global conformal symmetries Q acting far from the boundary, then we can conclude that the limiting theory in the infrared has to be conformal, with a finite boundary entropy. In such cases (when locality is preserved all the way to the far infrared) the boundary entropy stays bounded from below.

5. Proof of the gradient formula in the real time formalism

Here, we will use the machinery of real time spectral analysis for equilibrium boundary quantum field theory in 1 + 1 dimensions, as developed in [5]. Using the real time formalism, we will re-state the proof that $\mathrm{d}s/\mathrm{d}T \geq 0$ and the proof of the gradient formula for the boundary beta function, $\partial s/\partial\lambda^a = -g_{ab}\beta^b$. The Riemannian metric on the space of boundary couplings, $g_{ab}(\lambda)$, is $\chi_{ab}(0)/T$, the renormalized static susceptibility matrix of the boundary, divided by temperature. The dynamic susceptibility matrix of the boundary, $\chi_{ab}(\omega)$, is renormalized by natural subtractions in such a way that the static susceptibility matrix, $\chi_{ab}(0)$, remains positive.

The first step will be to show that

$$\frac{\mathrm{d}s}{\mathrm{d}T} = \frac{1}{2} T^{-2}\, \mathrm{Im}\, F'(0) \tag{48}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle \mathrm{i}[T_L(vt) + T_R(vt), \theta(0)]\rangle_{\mathrm{eq}}. \tag{49}$$

This is a Kubo formula for the change in the boundary entropy in response to a local change in the temperature at the boundary. The response function $F(\omega)$ is analytic in the upper half-plane. On the real axis, $-\mathrm{Re}\, F(\omega) \geq 0$.

The second step is to show that $\mathrm{Im}\, F'(0) \geq 0$, and therefore $\mathrm{d}s/\mathrm{d}T \geq 0$, by deriving a dispersion formula for $\mathrm{Im}\, F(\omega)$ in terms of $\mathrm{Re}\, F(\omega)$. The naive, unsubtracted dispersion formula is divergent, because $F(\omega)$ can grow as fast as ω for large ω, by canonical dimensional analysis in the ultraviolet limit. Fortunately, bulk conformal invariance will imply a vanishing formula, $F(\mathrm{i}2\pi T) = 0$, which gives a natural subtraction point. The subtracted dispersion formula converges as long as the ultraviolet behaviour is canonical (as long as the system approaches a renormalization group fixed point in the ultraviolet). The subtracted dispersion formula will still imply $\mathrm{Im}\, F'(0) \geq 0$, and therefore $\mathrm{d}s/\mathrm{d}T > 0$.

Equations (48), (49) were derived in [5] by considering the flow of entropy in real time, in and out of the boundary, in analogy with the flow of electric charge in an electric circuit. The flow of entropy is described by an entropy current operator, which is just the energy current divided by the temperature. The right-moving entropy current operator is the right-moving energy current divided by temperature, $j_L(x,t) = T_L(x,t)/T$. The boundary entropy 'charge' operator is $q_S(t) = -\theta(t)/T$. The Kubo formula for the entropic 'admittance' of the boundary was written, using the chirality of the bulk entropy currents, as

$$Y_S(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle \mathrm{i}[j_L(0,t), -q_S(0)]\rangle_{\mathrm{eq}}. \tag{50}$$

The entropic 'capacitance' of the boundary is

$$\frac{\mathrm{d}s}{\mathrm{d}T} = \lim_{\omega\to 0} \frac{1}{\mathrm{i}\omega} Y_S(\omega) = \mathrm{Im}\, Y_S'(0). \tag{51}$$

These are exactly equations (48), (49), since $T^{-2}F(\omega)/2 = Y_S(\omega)$. Here, we derive equations (48) and (49) directly.

The proof is based on the following assumptions:

(1) There is a local, symmetric energy–momentum tensor $T^{\mu}_{\nu}(x,t)$, $T_{\mu\nu} = T_{\nu\mu}$.

(2) The system is locally scale invariant in the bulk. The trace of the energy–momentum tensor vanishes in the bulk, $T^{\mu}_{\mu} = 0$.

(3) The bulk system is conformally invariant. The bulk ground state in the 'closed' channel is annihilated by the Virasoro operators $L_0 + \bar{L}_0$ and $L_1 + \bar{L}_1$.

(4) The system exhibits canonical scaling behaviour in the ultraviolet (goes to a renormalization group fixed point in the ultraviolet).

(5) The system is in equilibrium at temperature T. Equilibrium expectation values of commutators of local operators, $\langle [\mathcal{O}_1(t_1), \mathcal{O}_2(t_2)]\rangle_{\mathrm{eq}}$, go to zero at large times, $t_1 - t_2 \to \pm\infty$.

(6) The Fourier transforms

$$\int \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t} \langle [\mathcal{O}_1(t_1), \mathcal{O}_2(t_2)]\rangle_{\mathrm{eq}}$$

are smooth functions of the frequency ω, for any local operators $\mathcal{O}_1(t)$, $\mathcal{O}_2(t)$.

5.1. The Kubo formula for $\mathrm{d}s/\mathrm{d}T$

The bulk energy density operator is $[v^{-1}T_R(x-vt) + v^{-1}T_L(x+vt)]$ and the boundary energy operator is $-\theta(t)$, so the thermodynamic energy of the full system (of length L) is

$$-\frac{\partial}{\partial\beta} \ln Z = \langle H\rangle_{\mathrm{eq}} = \left\langle -\theta(t) + \int_0^L \mathrm{d}x\ [v^{-1}T_R(x-vt) + v^{-1}T_L(x+vt)] \right\rangle_{\mathrm{eq}}.$$

The equilibrium expectation values $\langle T_R(x-vt)\rangle_{\mathrm{eq}}$ and $\langle T_L(x+vt)\rangle_{\mathrm{eq}}$ are constant in x because they are independent of time, so $\langle v^{-1}T_R(x-vt) + v^{-1}T_L(x+vt)\rangle_{\mathrm{eq}}$ is the bulk

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energy density, $c\pi L/6\beta^2$, which is determined by bulk conformal invariance up to the value of the bulk conformal central charge, c. The difference between the total thermodynamic energy and the bulk energy is the thermodynamic boundary energy:

$$-\frac{\partial}{\partial\beta}\ln z = \langle -\theta(t)\rangle_{\mathrm{eq}}.$$

The boundary entropy is given by formula (13). Thus we have

$$T^2\frac{\partial s}{\partial T} = -\frac{\partial s}{\partial\beta} = \beta\frac{\partial}{\partial\beta}\langle\theta(0)\rangle_{\mathrm{eq}} = -\beta\langle H\,\theta(0)\rangle_{\mathrm{c}}. \tag{52}$$

Approximate the Hamiltonian by introducing an arbitrary cut-off point $x_1 > 0$:

$$H(x_1,t) = -\theta(t) + \int_0^{x_1}\mathrm{d}x\,\left[v^{-1}T_R(x-vt) + v^{-1}T_L(x+vt)\right].$$

Approximate βH by integrating over imaginary time, $\tau = \mathrm{i}t$, from 0 to β:

$$\beta H \approx \int_0^{-\mathrm{i}\beta}\mathrm{d}t\,\mathrm{i}H(x_1,t).$$

Then

$$T^2\frac{\partial s}{\partial T} = \lim_{x_1\to\infty}\int_0^{-\mathrm{i}\beta}\mathrm{d}t\,(-\mathrm{i})\langle H(x_1,t)\,\theta(0)\rangle_{\mathrm{c}}.$$

In fact, there is no dependence on x_1, because

$$\frac{\partial}{\partial x_1}\int_0^{-\mathrm{i}\beta}\mathrm{d}t\,\langle H(x_1,t)\,\theta(0)\rangle_{\mathrm{c}} = \int_0^{-\mathrm{i}\beta}\mathrm{d}t\,\langle[v^{-1}T_R(x_1-vt) + v^{-1}T_L(x_1+vt)]\,\theta(0)\rangle_{\mathrm{c}}$$

which is zero because the rhs, evaluated in the 'closed' channel where x is imaginary time, is a matrix element of the Virasoro operator $L_0 + \bar{L}_0$ between a boundary state and the bulk ground state, and the bulk ground state is annihilated by $L_0 + \bar{L}_0$. Therefore, for any $x_1 > 0$,

$$T^2\frac{\partial s}{\partial T} = \int_0^{-\mathrm{i}\beta}\mathrm{d}t\,(-\mathrm{i})\langle H(x_1,t)\,\theta(0)\rangle_{\mathrm{c}}. \tag{53}$$

Now deform the contour of integration in the standard way to obtain the Kubo formula:

$$\begin{aligned}\frac{\partial s}{\partial T} &= T^{-2}\left(\int_{-\infty-\mathrm{i}\beta}^{0-\mathrm{i}\beta} - \int_{-\infty}^{0}\right)\mathrm{d}t\,(-\mathrm{i})\langle H(x_1,t)\,\theta(0)\rangle_{\mathrm{c}}\\ &= T^{-2}\int_{-\infty}^{0}\mathrm{d}t\,\langle\mathrm{i}[H(x_1,t),\theta(0)]\rangle_{\mathrm{eq}}.\end{aligned} \tag{54}$$

This is the Kubo formula for the entropic 'capacitance' of the boundary, which was derived in [5] as the infinitesimal change in the entropic 'charge,' $-\theta(t)/T$, produced in real time by an infinitesimal change in the entropic 'potential' of the boundary.

The integrand in the Kubo formula is a distribution in t. In the derivation, the contour deformation in the complex time plane is justified by Gauss's law for distributions, applied on the region $-\mathrm{i}\beta \le \mathrm{Im}\,t \le 0$, $\mathrm{Re}\,t \le 0$. The integral over the boundary of this region vanishes. The boundary integral can be separated unambiguously into two parts—the integral over the imaginary t axis from 0 to $-\mathrm{i}\beta$, and the rest—because the integrand is an ordinary function near $t = 0$ and near $t = -\mathrm{i}\beta$. In general, equilibrium expectation values satisfy

$$\langle H(x_1, t - \mathrm{i}\beta)\, \theta(0)\rangle_{\mathrm{eq}} = \langle \theta(0)\, H(x_1, t)\rangle_{\mathrm{eq}}$$

but here,

$$\langle [H(x_1, t), \theta(0)]\rangle_{\mathrm{eq}} = \langle [H, \theta(0)]\rangle_{\mathrm{eq}} = 0$$

for all real t in the range $-x_1 < vt < x_1$, by causality. It takes at least time x_1/v for any effect of the cut-off at x_1 to reach the boundary, or vice versa. Therefore the integrand in the Kubo formula is identically zero near $t = 0$. The equilibrium correlation function, $\langle H(x_1, t)\, \theta(0)\rangle_{\mathrm{eq}}$, is periodic on the imaginary t axis, with period $\mathrm{i}\beta$, without singularity at $t = 0$ or $t = \mathrm{i}\beta$.

A second Kubo formula is obtained by deforming the integration contour in equation (53) to positive times:

$$\frac{\partial s}{\partial T} = T^{-2} \int_0^{\infty} \mathrm{d}t\, \langle -\mathrm{i}[H(x_1, t), \theta(0)]\rangle_{\mathrm{eq}}. \tag{55}$$

5.2. Using chirality of the energy currents

Local conservation of energy implies

$$\partial_t H(x_1, t) = -T_R(x_1 - vt) + T_L(x_1 + vt),$$

and thus

$$T^2 \frac{\partial s}{\partial T} = \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t'\, \langle \mathrm{i}[-T_R(x_1 - vt') + T_L(x_1 + vt'), \theta(0)]\rangle_{\mathrm{eq}}. \tag{56}$$

The boundary term at $t' = -\infty$ can be neglected because equilibrium expectation values of commutators of local operators decay to zero at large times. Now use the chirality of the bulk energy currents. For all $t' < x_1/v$,

$$\langle [T_R(x_1 - vt'), \theta(0)]\rangle_{\mathrm{eq}} = \langle [T_R(x_1 - vt', 0), \theta(0)]\rangle_{\mathrm{eq}} = 0$$

as an equal-time commutator of spatially separated operators. Therefore

$$\begin{aligned} T^2 \frac{\partial s}{\partial T} &= \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t'\; \langle \mathrm{i}[T_L(x_1 + vt'), \theta(0)]\rangle_{\mathrm{eq}} \\ &= \int_{-\infty}^{0} \mathrm{d}t' \int_{t'}^{0} \mathrm{d}t\; \langle \mathrm{i}[T_L(x_1 + vt'), \theta(0)]\rangle_{\mathrm{eq}} \\ &= \int_{-\infty}^{0} \mathrm{d}t'\; (-t') \langle \mathrm{i}[T_L(x_1 + vt'), \theta(0)]\rangle_{\mathrm{eq}} \\ &= \int_{-\infty}^{\infty} \mathrm{d}t'\; (-t') \langle \mathrm{i}[T_L(x_1 + vt'), \theta(0)]\rangle_{\mathrm{eq}}. \end{aligned} \tag{57}$$

In the last step, it makes no difference to extend the time integral to $+\infty$, because, for all $t' > -x_1/v$,

$$\langle [T_L(x_1 + vt'), \theta(0)]\rangle_{\mathrm{eq}} = \langle [T_L(x_1 + vt', 0), \theta(0)]\rangle_{\mathrm{eq}} = 0$$

as an equal-time commutator of spatially separated operators. Next, change the integration variable to $t = t' - x_1/v$, obtaining

$$T^2 \frac{\partial s}{\partial T} = \int_{-\infty}^{\infty} \mathrm{d}t\,(-t + x_1/v)\langle \mathrm{i}[T_L(vt), \theta(0)]\rangle_{\mathrm{eq}}.$$

The term proportional to x_1 vanishes by (23) and thus

$$T^2 \frac{\partial s}{\partial T} = \int_{-\infty}^{\infty} \mathrm{d}t\,(-t)\,\langle \mathrm{i}[T_L(0,t), \theta(0)]\rangle_{\mathrm{eq}}. \tag{58}$$

In terms of the response function

$$F_L(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t}\langle \mathrm{i}[T_L(0,t), \theta(0)]\rangle_{\mathrm{eq}}, \tag{59}$$

$$T^2 \frac{\partial s}{\partial T} = \mathrm{i}^{-1} F_L'(0). \tag{60}$$

$F_L(\omega)$ is analytic in the upper half-plane because the commutator vanishes for all $t > 0$, by the chirality of the energy current.

By similar reasoning, the second Kubo formula, equation (55), becomes

$$T^2 \frac{\partial s}{\partial T} = \int_{-\infty}^{\infty} \mathrm{d}t\, t\langle -\mathrm{i}[T_R(0,t), \theta(0)]\rangle_{\mathrm{eq}}. \tag{61}$$

In terms of the response function

$$F_R(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\langle -\mathrm{i}[T_R(0,t), \theta(0)]\rangle_{\mathrm{eq}}, \tag{62}$$

$$T^2 \frac{\partial s}{\partial T} = \mathrm{i}^{-1} F_R'(0). \tag{63}$$

$F_R(\omega)$ is analytic in the upper half-plane because the commutator vanishes for all $t < 0$, by the chirality of the energy current. Finally, define

$$\begin{aligned} F(\omega) &= F_L(\omega) + F_R(\omega) \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t}\langle \mathrm{i}[T_L(vt) - T_R(vt), \theta(0)]\rangle_{\mathrm{eq}} \end{aligned} \tag{64}$$

so

$$T^2 \frac{\partial s}{\partial T} = \frac{1}{2}\mathrm{i}^{-1} F'(0). \tag{65}$$

5.3. Properties of $F(\omega)$

(1) $F(\omega)$ is analytic in the upper half-plane.
(2) $\overline{F(\omega)} = F(-\bar{\omega})$.
(3) $F(0) = 0$.
(4) $T^2(\partial s/\partial T) = \frac{1}{2}\,\mathrm{Im}\, F'(0)$.
(5) $F(\omega)/\omega^2 \to 0$ as $\omega \to \pm\infty$.
(6) $\mathrm{Re}\, F(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t}\omega\langle[\theta(t), \theta(0)]\rangle_{\mathrm{eq}}$.
(7) $-\mathrm{Re}\, F(\omega) \geq 0$ for all real ω.
(8) $F(\mathrm{i}2\pi T) = 0$.

$F(\omega)$ is analytic in the upper half-plane because both $F_L(\omega)$ and $F_R(\omega)$ are. $\overline{F(\omega)} = F(-\bar{\omega})$ because $T_L(x,t)$, $T_R(x,t)$ and $\theta(t)$ are self-adjoint operators. Property (2) implies that $F'(0)$ is imaginary, so $T^2\partial s/\partial T = \mathrm{Im}\, F'(0)/2$. $F(0) = 0$ by equation (23). $F(\omega)/\omega^2 \to 0$ as $\omega \to \infty$ by canonical dimensional analysis in the ultraviolet limit. Each of $T_L(x,t)$ and $T_R(x,t)$ has scaling dimension 2, while $\theta(t)\mathrm{d}t = \beta^a\phi_a(t)\mathrm{d}t$ goes to zero in the ultraviolet limit, assuming that the system goes to a renormalization group fixed point in the ultraviolet. For property (6), evaluate

$$\begin{aligned}\mathrm{Re}\, F(\omega) &= \mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t}\langle \mathrm{i}[T_L(0,t) - T_R(0,t), \theta(0)]\rangle_{\mathrm{eq}} \\ &= \mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t}\langle \mathrm{i}[-\theta'(t), \theta(0)]\rangle_{\mathrm{eq}} \\ &= \int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t}\omega\langle [\theta(t), \theta(0)]\rangle_{\mathrm{eq}}.\end{aligned}$$

For property (7), define the operator Fourier modes

$$\tilde{\theta}(\omega) = \int_{-\infty}^{\infty}\mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\theta(t)$$

satisfying

$$\begin{aligned}&\tilde{\theta}(\omega)^\dagger = \tilde{\theta}(-\omega) \\ &[H, \tilde{\theta}(\omega)] = -\omega\tilde{\theta}(\omega).\end{aligned}$$

Then we have

$$2\pi\delta(\omega' + \omega)\mathrm{Re}\,\omega^{-1}F(\omega) = \langle [\tilde{\theta}(\omega'), \tilde{\theta}(\omega)]\rangle_{\mathrm{eq}}. \tag{66}$$

Next note that

$$\langle [\tilde{\theta}(\omega'), \tilde{\theta}(\omega)]\rangle_{\mathrm{eq}} = \left(1 - \mathrm{e}^{\beta\omega}\right)\langle \tilde{\theta}(\omega')\,\tilde{\theta}(\omega)\rangle_{\mathrm{eq}}$$

and thus

$$2\pi\delta(\omega' + \omega)\mathrm{Re}\,\omega^{-1}F(\omega) = \left(1 - \mathrm{e}^{\beta\omega}\right)\langle \tilde{\theta}(\omega')\,\tilde{\theta}(\omega)\rangle_{\mathrm{eq}} \tag{67}$$

which implies

$$-\mathrm{Re}\, F(\omega) \geq 0. \tag{68}$$

Finally, for property (8), write

$$\langle [T_L(0,t), \theta(0)]\rangle_{\mathrm{eq}} = \frac{1}{2\pi}\int \mathrm{d}\omega\, \langle [T_L(0,t), \tilde{\theta}(\omega)]\rangle_{\mathrm{eq}}$$

so

$$\begin{aligned}&\langle [T_L(0,t), \tilde{\theta}(\omega)]\rangle_{\mathrm{eq}} = \mathrm{e}^{\mathrm{i}\omega t}(-\mathrm{i})F_L(\omega) \\ &\left(1 - \mathrm{e}^{\beta\omega}\right)\langle T_L(0,t)\,\tilde{\theta}(\omega)\rangle = \langle [T_L(0,t), \tilde{\theta}(\omega)]\rangle_{\mathrm{eq}} \\ &\langle T_L(0,t)\,\tilde{\theta}(\omega)\rangle_{\mathrm{c}} = \mathrm{e}^{\mathrm{i}\omega t}\left(1 - \mathrm{e}^{\beta\omega}\right)^{-1}(-\mathrm{i})F_L(\omega)\end{aligned}$$

and therefore

$$\langle T_L(0,t)\,\theta(0)\rangle_{\mathrm{c}} = \frac{1}{2\pi\mathrm{i}}\int \mathrm{d}\omega\ \mathrm{e}^{\mathrm{i}\omega t}\frac{F_L(\omega)}{1 - \mathrm{e}^{\beta\omega}}. \tag{69}$$

We next Wick rotate to imaginary time $\tau = \mathrm{i}t$, for $0 < \tau < \beta$, to get

$$\langle T_L(0, -\mathrm{i}\tau)\,\theta(0)\rangle_{\mathrm{c}} = \frac{1}{2\pi\mathrm{i}} \int \mathrm{d}\omega\, \mathrm{e}^{\omega\tau} \frac{F_L(\omega)}{1 - \mathrm{e}^{\beta\omega}}. \tag{70}$$

Deform the contour of integration into the upper half-plane, picking up the residues at the thermal poles:

$$\langle T_L(0, -\mathrm{i}\tau)\,\theta(0)\rangle_{\mathrm{c}} = \frac{-1}{\beta} \sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}\omega_n\tau} F_L(\mathrm{i}\omega_n) \tag{71}$$

where

$$\omega_n = \frac{2\pi n}{\beta}.$$

Then, by chirality of the energy current,

$$\langle T_L(x - \mathrm{i}v\tau)\,\theta(0)\rangle_{\mathrm{c}} = \frac{-1}{\beta} \sum_{n=1}^{\infty} \mathrm{e}^{-\omega_n(x - \mathrm{i}v\tau)/v} F_L(\mathrm{i}\omega_n). \tag{72}$$

Similarly, also for $0 < \tau < \beta$,

$$\langle T_R(0, -\mathrm{i}\tau)\,\theta(0)\rangle_{\mathrm{c}} = \frac{1}{2\pi\mathrm{i}} \int \mathrm{d}\omega\, \mathrm{e}^{-\omega\tau} \frac{F_R(\omega)}{\mathrm{e}^{-\beta\omega} - 1} \tag{73}$$

so

$$\langle T_R(x + \mathrm{i}v\tau)\,\theta(0)\rangle_{\mathrm{c}} = \frac{-1}{\beta} \sum_{n=1}^{\infty} \mathrm{e}^{-\omega_n(x + \mathrm{i}v\tau)/v} F_R(\mathrm{i}\omega_n). \tag{74}$$

Setting $n = 1$,

$$F(\mathrm{i}\omega_1) = \frac{2\pi}{\beta} \langle K_1(x)\,\theta(0)\rangle_{\mathrm{eq}} \tag{75}$$

where

$$K_1(x) = \frac{-\beta}{2\pi v} \int_0^{v\beta} \mathrm{d}y\, \left[\mathrm{e}^{\omega_1(x - \mathrm{i}y)/v} T_L(x - \mathrm{i}y) + \mathrm{e}^{\omega_1(x + \mathrm{i}y)/v} T_R(x + \mathrm{i}y)\right]. \tag{76}$$

In the 'closed' channel, where x is imaginary time, $K_1(x)$ is the Virasoro operator $L_1 + \bar{L}_1$. Therefore, in the 'closed' channel, $F(\mathrm{i}\omega_1)$ is a matrix element of $L_1 + \bar{L}_1$ between a boundary state and the bulk ground state. Global conformal invariance of the bulk system implies that $L_1 + \bar{L}_1$ annihilates the bulk ground state in the 'closed' channel. Therefore $F(\mathrm{i}\omega_1) = F(2\pi\mathrm{i}/\beta) = 0$.

5.4. Subtracted dispersion formula for $\mathrm{Im}\, F'(0)$

The vanishing formulae, $F(0) = F(2\pi\mathrm{i}/\beta) = 0$, allow writing

$$\eta^{-1} F(\eta) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\omega \left[\frac{1}{\omega - \eta - \mathrm{i}\epsilon} - \frac{\omega + \eta + \mathrm{i}\epsilon}{\omega^2 + \omega_1^2}\right] \omega^{-1} F(\omega).$$

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The integral converges, because $F(\omega)/\omega^2 \to 0$ when $\omega \to \pm\infty$. Take the imaginary part to get the dispersion formula

$$\operatorname{Im} \eta^{-1} F(\eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \left[\mathcal{P}\left(\frac{1}{\omega - \eta}\right) - \frac{\omega + \eta}{\omega^2 + \omega_1^2} \right] \left[-\omega^{-1} \operatorname{Re} F(\omega) \right]. \quad (77)$$

Take $\eta \to 0$ to get

$$\operatorname{Im} F'(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{-\operatorname{Re} F(\omega)}{\omega^2 (1 + \omega_1^{-2} \omega^2)}. \quad (78)$$

Thus $\operatorname{Im} F'(0) \geq 0$. Equality, $\operatorname{Im} F'(0) = 0$, is possible only if $\operatorname{Re} F(\omega) = 0$, which implies $F(\omega) = 0$. It follows then from equation (67) that $\tilde{\theta}(\omega)$ is proportional to $\delta(\omega)$, which implies that $\theta(t)$ is a multiple of the identity. This means that the boundary field theory is scale invariant. Therefore $\partial s/\partial T \geq 0$, with equality if and only if the boundary field theory is scale invariant.

5.5. The gradient formula

Calculate

$$\begin{aligned}
\partial_a s &= \partial_a \left(1 - \beta \frac{\partial}{\partial \beta}\right) z \\
&= \left(1 - \beta \frac{\partial}{\partial \beta}\right) \partial_a z \\
&= \left(1 - \beta \frac{\partial}{\partial \beta}\right) \langle \beta \phi_a(0) \rangle_{\mathrm{eq}} \\
&= -\beta^2 \frac{\partial}{\partial \beta} \langle \phi_a(0) \rangle_{\mathrm{eq}} \\
&= \beta^2 \langle H \, \phi_a(0) \rangle_{\mathrm{c}} \\
&= \beta \int_0^{-\mathrm{i}\beta} \mathrm{d}t \, \mathrm{i} \langle H(x_1, t) \, \phi_a(0) \rangle_{\mathrm{c}}
\end{aligned} \quad (79)$$

where the last expression is independent of x_1 and the result is thus exact, as before.

Deform the integration contour to negative times to get

$$\begin{aligned}
-T \partial_a s &= \left(\int_{-\infty}^{0} - \int_{-\infty - \mathrm{i}\beta}^{0 - \mathrm{i}\beta} \right) \mathrm{d}t \, \mathrm{i} \langle H(x_1, t) \, \phi_a(0) \rangle_{\mathrm{c}} \\
&= \int_{-\infty}^{0} \mathrm{d}t \, \langle \mathrm{i}[H(x_1, t), \phi_a(0)] \rangle_{\mathrm{c}} \\
&= \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \, \langle \mathrm{i}[T_L(x_1, t') - T_R(x_1, t'), \phi_a(0)] \rangle_{\mathrm{c}} \\
&= \int_{-\infty}^{0} \mathrm{d}t \, (-t) \langle \mathrm{i}[T_L(x_1, t), \phi_a(0)] \rangle_{\mathrm{c}} \\
&= \mathrm{i}^{-1} F'_{\mathrm{La}}(0)
\end{aligned} \quad (80)$$

where

$$F_{\mathrm{La}}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{-\mathrm{i}\omega t} \langle \mathrm{i}[T_L(0, t), \phi_a(0)] \rangle_{\mathrm{eq}}. \quad (81)$$

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The spectral function $F_{\mathrm{La}}(\omega)$ is analytic in the upper half-plane. Equation (81) is a Kubo formula giving the response of the boundary fields to a local change of temperature.

A second Kubo formula is obtained similarly by deforming the contour to positive times:

$$-T\partial_a s = \mathrm{i}^{-1} F'_{\mathrm{Ra}}(0) \tag{82}$$

where

$$F_{\mathrm{Ra}}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \langle (-i)[T_R(0,t), \phi_a(0)]\rangle_{\mathrm{eq}}. \tag{83}$$

$F_{\mathrm{Ra}}(\omega)$ is analytic in the upper half-plane. Define

$$\begin{aligned} F_a(\omega) &= F_{\mathrm{La}}(\omega) + F_{\mathrm{Ra}}(\omega) \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle \mathrm{i}[T_L(vt) - T_R(vt), \phi_a(0)]\rangle_{\mathrm{eq}} \end{aligned} \tag{84}$$

so

$$-T\partial_a s = \tfrac{1}{2}\mathrm{i}^{-1} F'_a(0). \tag{85}$$

$F_a(\omega)$ satisfies

(1) $F_a(\omega)$ is analytic in the upper half-plane.
(2) $\overline{F_a(\omega)} = F_a(-\bar{\omega})$.
(3) $F_a(0) = 0$.
(4) $-T\partial_a s = \frac{1}{2}\,\mathrm{Im}\, F'_a(0)$.
(5) $F_a(\mathrm{i}2\pi T) = 0$.
(6) $\beta^a F_a(\omega) = F(\omega)$.
(7) $\mathrm{Re}\, F_a(\omega) = \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \omega \langle [\theta(t), \phi_a(0)]\rangle_{\mathrm{eq}}$.
(8) $\mathrm{Re}\, F_a(\omega)/\omega^2 \to 0$ as $\omega \to \infty$.

Properties (1)–(5) are derived just as for $F(\omega)$. Property (6) follows from $\theta(t) = \beta^a \phi_a(t)$. For property (7), evaluate

$$\begin{aligned} \mathrm{Re}\, F_a(\omega) &= \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle \mathrm{i}[T_L(0,t) - T_R(0,t), \phi_a(0)]\rangle_{\mathrm{eq}} \\ &= \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle \mathrm{i}[-\theta'(t), \phi_a(0)]\rangle_{\mathrm{eq}} \\ &= \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \omega \langle [\theta(t), \phi_a(0)]\rangle_{\mathrm{eq}}. \end{aligned} \tag{86}$$

Property (8) follows from property (7) and canonical scaling in the ultraviolet, since the boundary operators, $\phi_a(t)$, all have scaling dimensions ≤ 1, and $\theta(t)\,\mathrm{d}t$ vanishes in the ultraviolet limit.

As before, the vanishing formulae allow writing a subtracted dispersion formula:

$$\mathrm{Im}\, \eta^{-1} F_a(\eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \left[\mathcal{P}\left(\frac{1}{\omega - \eta}\right) - \frac{\omega + \eta}{\omega^2 + \omega_1^2} \right] \left[-\omega^{-1}\, \mathrm{Re}\, F_a(\omega)\right] \tag{87}$$

where $\omega_1 = 2\pi T$. Take $\eta \to 0$ to get

$$\mathrm{Im}\, F'_a(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{-\mathrm{Re}\, F_a(\omega)}{\omega^2(1+\omega_1^{-2}\omega^2)}. \tag{88}$$

Now use property (7) and the identity $\theta(t) = \beta^a \phi_a(t)$ to write

$$-\mathrm{Re}\, F_a(\omega) = \mathrm{Re}\, \omega G_{ab}(\omega)\beta^b \tag{89}$$

where

$$G_{ab}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \langle [-\phi_b(t), \phi_a(0)] \rangle_{\mathrm{eq}}. \tag{90}$$

In terms of the operator Fourier modes

$$\tilde{\phi}_a(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \phi_a(t),$$

$$\langle [-\tilde{\phi}_b(\omega'), \tilde{\phi}_a(\omega)] \rangle_{\mathrm{eq}} = 2\pi\delta(\omega' + \omega) G_{ab}(\omega)$$

so

$$\langle \tilde{\phi}_b(\omega')\, \tilde{\phi}_a(\omega) \rangle_{\mathrm{c}} = 2\pi\delta(\omega' + \omega) \frac{G_{ab}(\omega)}{\mathrm{e}^{\beta\omega} - 1}. \tag{91}$$

Therefore

(1) $G_{ab}(\omega)/\omega$ is a non-negative Hermitian matrix,

(2) $G_{ab}(-\omega) = -\overline{G_{ab}(\omega)}$,

(3) $G_{ab}(-\omega) = -G_{ba}(\omega)$.

The dispersion formula for $\mathrm{Im}\, F'_a(0)$ becomes

$$\mathrm{Im}\, F'_a(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{\mathrm{Re}\, G_{ab}(\omega)\beta^b}{\omega(1+\omega_1^{-2}\omega^2)} \tag{92}$$

which is the gradient formula,

$$\partial_a s = -g_{ab}\beta^b$$

with

$$T g_{ab} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{\mathrm{Re}\, G_{ab}(\omega)}{\omega(1+\omega_1^{-2}\omega^2)} \tag{93}$$

being a positive symmetric matrix, a Riemannian metric on the space of boundary couplings.

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5.6. The renormalized boundary susceptibility matrix

Formally, the dynamic susceptibility matrix is given by the Kubo formula

$$\chi_{ab}(\omega) = \int_{-\infty}^{0} \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t} \langle \mathrm{i}[-\phi_b(t), \phi_a(0)]\rangle_{\mathrm{eq}} \tag{94}$$

however the integral diverges at $t=0$. The unrenormalized boundary susceptibilities are divergent. The Fourier transform of the formal Kubo formula is

$$\chi_{ab}(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{G_{ab}(\omega)}{\omega-\eta-\mathrm{i}\epsilon} \tag{95}$$

which diverges, in general, since $G_{ab}(\omega)$ can grow as fast as ω for large ω. Renormalizing the boundary susceptibilities requires two subtractions: a constant subtraction and a linear subtraction, proportional to η. A renormalized dynamic susceptibility matrix is defined by

$$\chi^{ren}_{ab}(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \left[\frac{1}{\omega-\eta-\mathrm{i}\epsilon} - \frac{\omega+\eta+\mathrm{i}\epsilon}{\omega^2+\omega_1^2}\right] G_{ab}(\omega). \tag{96}$$

The subtractions are chosen so that $\chi^{ren}_{ab}(\omega)$ will be compatible with the natural susceptibilities $F_a(\omega)$ and $F(\omega)$:

$$F_a(\omega) = \chi_{ab}(\omega)\beta^b$$
$$F(\omega) = \chi_{ab}(\omega)\beta^a\beta^b.$$

$F_a(\omega)$ and $F(\omega)$ are natural in the sense that they are constructed without arbitrary subtractions, in terms of the chiral energy currents outside the boundary. $\chi^{ren}_{ab}(\omega)$ is a dynamic susceptibility matrix in the sense that (1) it is analytic in the upper half-plane, (2) it satisfies, on the real axis,

$$\chi^{ren}_{ab}(\omega) - \overline{\chi^{ren}_{ba}(\omega)} = \mathrm{i}G_{ab}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t}\langle \mathrm{i}[-\phi_b(t), \phi_a(0)]\rangle_{\mathrm{eq}}$$

and (3) its static limit,

$$\chi^{ren}_{ab}(0) = T g_{ab}, \tag{97}$$

is a positive symmetric matrix. The metric g_{ab} on the space of boundary couplings is the renormalized static susceptibility matrix for the boundary, divided by the temperature.

5.7. The imaginary time formula for the metric

The imaginary time formula for the metric g_{ab} is [3]

$$T g_{ab} = \int_{0}^{\beta} \mathrm{d}\tau\, \langle \phi_b(-\mathrm{i}\tau)\, \phi_a(0)\rangle_{\mathrm{c}} \left[1-\cos(\omega_1\tau)\right]. \tag{98}$$

From equation (91),

$$\langle \phi_b(-\mathrm{i}\tau)\, \phi_a(0)\rangle_{\mathrm{c}} = \frac{1}{2\pi}\int \mathrm{d}\omega\, \mathrm{e}^{\omega\tau}\frac{G_{ab}(\omega)}{\mathrm{e}^{\beta\omega}-1}$$

and therefore

$$Tg_{ab} = \frac{1}{2\pi}\int \mathrm{d}\omega\, \frac{G_{ab}(\omega)}{\mathrm{e}^{\beta\omega}-1}\int_0^\beta \mathrm{d}\tau\, \mathrm{e}^{\omega\tau}\left[1-\frac{1}{2}\mathrm{e}^{\mathrm{i}\omega_1\tau}-\frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega_1\tau}\right]$$
$$= \frac{1}{2\pi}\int \mathrm{d}\omega\, \frac{G_{ab}(\omega)}{\omega(1+\omega_1^{-2}\omega^2)}$$

which is exactly equation (93) for Tg_{ab}, since $G_{ab}(-\omega) = -\overline{G_{ab}(\omega)}$. So the real time and imaginary time formulae for the metric g_{ab} are equivalent.

6. Estimate of $\mathrm{d}s/\mathrm{d}T$ using the dispersion formula

Combining formula (65) with the dispersion relation (78) we can write

$$\frac{\mathrm{d}s}{\mathrm{d}T} = 2\pi\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{-\mathrm{Re}\, F(\omega,T)}{\omega^2(4\pi^2T^2+\omega^2)}. \tag{99}$$

Here we included the temperature argument in the notation for the response function: $F(\omega) = F(\omega, T)$. By property (2) from section 5.3 the function $\mathrm{Re}\, F(\omega, T)$ is even on the real axis. Since $F(0, T) = 0$ (property (3)) the integral on the right-hand side of (99) is well defined for $T > 0$. In the limit $T \to 0$ however the poles at $\omega = 0$ and $\omega = \pm\mathrm{i}\omega_1$ coalesce. To separate off the dangerous part we rewrite the right-hand side as

$$\frac{\mathrm{d}s}{\mathrm{d}T} = 2\pi\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{-\mathrm{Re}\, F(\omega,T)+(\omega^2/2)\,\mathrm{Re}\, F''(0,T)}{\omega^2(4\pi^2T^2+\omega^2)} - \frac{\pi}{2T}F''(0,T). \tag{100}$$

Here the integral on the right-hand side converges as $T \to 0$ to a constant

$$f = -2\pi\int_{-\infty}^{+\infty} \mathrm{d}\omega\ \mathrm{Re}\, F(\omega,0)\mathcal{P}\left(\frac{1}{\omega^4}\right) \tag{101}$$

where $\mathcal{P}(1/\omega^4)$ is the standard (even) distribution regularizing the function $1/\omega^4$. The first term in (100) is therefore integrable at $T = 0$.

Let us look now at the second term on the right-hand side of (100). By comparing formula (33) with the $T \to 0$ limit of formula (73) we obtain

$$A_{\theta R}(\omega) = -\lim_{T\to 0} F_R(\omega, T). \tag{102}$$

It follows then from (44) that

$$\lim_{T\to 0} F''(0,T) = 0. \tag{103}$$

However general analysis stops here as we have no control in general over how fast $F''(0, T)$ vanishes as $T \to 0$ and thus cannot conclude whether

$$\frac{\mathrm{Re}\, F''(0,T)}{T}$$

is integrable in a neighbourhood of $T = 0$.

Equation (103) implies that $T\mathrm{d}s/\mathrm{d}T \to 0$ as $T \to 0$. Note that in deriving this we used in the essential way the consequences of the bulk conformal invariance on a half-plane, that is the condition $\langle 0|Q = 0$ discussed in section 4.

Although the above manipulations do not lead to demonstrating the boundedness of s they boil the problem down to having an estimate of the zero-temperature limit of $F''(0,T)$. For models possessing an asymptotic power expansion at small temperature (such as the Ising model with a boundary magnetic field) the existence of a lower bound on s follows from (100), (103).

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Appendix A. $\langle\theta\theta\rangle_c$ decays at infinity from bulk conformal invariance at $T>0$

From (19) and (73) we obtain

$$\begin{aligned}
-\frac{2\pi}{\beta}\mathrm{e}^{-2\pi n x/v\beta}\langle 0|L_n\theta(0)|B\rangle &= \int_0^\beta \mathrm{d}\tau\,\mathrm{e}^{2\pi\mathrm{i}n\tau/\beta}\langle T_R(x+\mathrm{i}\tau)\theta(0)\rangle_{\mathrm{eq}} \\
&= \frac{1}{2\pi}\int_0^\beta \mathrm{d}\tau\,\mathrm{e}^{-2\pi\mathrm{i}n\tau/\beta}\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\mathrm{e}^{\omega(-\tau+\mathrm{i}x/v)}\frac{-\mathrm{i}F_R(\omega)}{(\mathrm{e}^{-\beta\omega}-1)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\omega x/v}\frac{\mathrm{i}F_R(\omega)}{(\omega-2\pi\mathrm{i}n/\beta)}
\end{aligned} \tag{A.1}$$

where on the left-hand side $\langle 0|$ is the conformal bulk vacuum, $|B\rangle$ is the boundary state representing our boundary condition on a cylinder of circumference β. Here we use a representation for correlators corresponding to quantization with x being a Euclidean time. Since $\langle 0|L_n=0$ for $n\le 1$, we have

$$\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\omega x/v}\frac{F_R(\omega)}{(\omega-2\pi\mathrm{i}n/\beta)}=0,\qquad n\le 1,\quad x>0. \tag{A.2}$$

Taking an appropriate linear combination of the above equations with $n=-1,0,1$ we get

$$\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\omega x/v}\frac{F_R(\omega)}{\omega(\omega^2+4\pi^2/\beta^2)}=0,\qquad x>0. \tag{A.3}$$

The limit $x\to 0$ gives a sum rule

$$\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\frac{F_R(\omega)}{\omega(\omega^2+4\pi^2/\beta^2)}=0. \tag{A.4}$$

As proved in section 5, the function $\mathrm{Re}\,F_R(\omega)$ is even on the real line and the function $\mathrm{Im}\,F_R(\omega)$ is odd. Thus equation (A.3) implies that

$$\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\frac{[\cos(\omega x/v)\,\mathrm{Im}\,F_R(\omega)+\sin(\omega x/v)\,\mathrm{Re}\,F_R(\omega)]}{\omega(\omega^2+4\pi^2/\beta^2)}=0 \tag{A.5}$$

for $x>0$. The spectral function $F_R(\omega)=F_R(\omega,T)$ is related to the zero-temperature spectral function as in (102)

$$\lim_{T\to 0}F_R(\omega,T)=-A_{\theta R}(\omega).$$

Thus in the limit $T \to 0$ equations (A.5) (assuming the limit commutes with the integration) imply that the functions $\mathrm{Im}\, A_{R\theta}(\omega)/\omega^3$ and $\mathrm{Re}\, A_{R\theta}(\omega)/\omega^2$ are integrable at $\omega = 0$.

We did not use here the analyticity of $F_R(\omega)$ in the upper half-plane. That and the sum rule (A.4) imply that $F_R(\mathrm{i}2\pi T) = 0$. This zero can be considered as the sole manifestation of the conformal invariance at $T > 0$. Another zero is situated at $\omega = 0$.

This allows us to write a subtracted dispersion relation

$$\begin{aligned} \frac{\mathrm{Re}\, F_R(\omega)}{(\omega^2 + 4\pi^2/\beta^2)} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\eta\, \frac{\mathrm{Im}\, F_R(\omega)}{(\eta^2 + 4\pi^2/\beta^2)} \mathcal{P}\left(\frac{1}{\eta - \omega}\right), \\ \frac{\mathrm{Im}\, F_R(\omega)}{(\omega^2 + 4\pi^2/\beta^2)} &= -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\eta\, \frac{\mathrm{Re}\, F_R(\omega)}{(\eta^2 + 4\pi^2/\beta^2)} \mathcal{P}\left(\frac{1}{\eta - \omega}\right). \end{aligned} \tag{A.6}$$

As $T \to 0$ these dispersion relations at least formally (more on this below) go into the dispersion relations (43) and the sum rule (A.4) goes into the sum rule (42). It was demonstrated in section 4 how the last two imply (44). Analogous considerations hold for $F_L(\omega)$ and imply (45).

It should be stressed that in all these manipulations it is implicitly assumed that taking the limit $T \to 0$ commutes with integrations in dispersion relations. Or equivalently that the dispersion relations for $T > 0$ go to those at $T = 0$ in a continuous fashion. It could happen that there are singularities of $F_R(\omega)$ in the lower half-plane that approach $\omega = 0$ as $T \to 0$. The above conclusions for the behaviour of $A_{\theta R}(\omega)$ at zero would then be incorrect. The continuity at $T = 0$ of the above equations is essentially equivalent to the condition of the asymptotic conformal invariance on a half-plane $\langle 0|Q = 0$ discussed in section 4.

To conclude we see that, assuming the just discussed continuity at $T = 0$, the finite T conformal invariance implies formulae (44), (45) and, as a consequence, the vanishing of the $\langle \theta(\tau)\theta(\tau')\rangle_c$ correlator in the infrared.

Appendix B. A lower bound on boundary entropy of Cardy states[3]

For rational conformal field theories there is a set of local conformal boundary conditions that preserve the chiral algebra in the most straightforward way—Cardy boundary states. These boundary states are constructed via the modular S-matrix S_{ij} as

$$|i\rangle = \sum_j \frac{S_{ij}}{\sqrt{S_{0j}}} |j\rangle\rangle$$

where $|j\rangle\rangle$ are Ishibashi states. (Note that in the proof of the Verlinde formula one shows at an intermediate step that $S_{0j} > 0$ and thus the division makes sense.) The boundary entropies of the Cardy states are

$$g_i = \langle 0|i\rangle = \frac{S_{i0}}{\sqrt{S_{00}}}.$$

We are going to show that

$$g_i \geq g_0 = \sqrt{S_{00}}. \tag{B.1}$$

[3] The contents of this appendix grew from discussions of AK with G Moore.

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The S-matrix entries S_{ij} can be considered as a collection of common eigenvectors of the fusion matrices. By the Perron–Frobenius theorem there is a unique a common eigenvector whose set of eigenvalues consists of the maximal eigenvalues of the respective fusion matrices. It is uniquely characterized by the property that all its entries are positive. Since $S_{0j} > 0$ this eigenvector corresponds to the zero weight and the corresponding collection of maximal eigenvalues is

$$\gamma_i^{\max} = \frac{S_{i0}}{S_{00}}.$$

On the other hand, the inequality (B.1) translates into

$$S_{i0} \geq S_{00}$$

and hence we need to prove that the maximal eigenvalues of the fusion matrices are all greater or equal to one. To this end we note that the dimension of the Friedan–Shenker vector bundle [10] over a genus g Riemann surface with n punctures in representations $i_1, \ldots, i_n$ is given by the formula [8]

$$\dim \mathcal{H}(\Sigma_g; (P_1, i_1), \ldots, (P_n, i_n)) = \sum_p \frac{1}{(S_{0p})^{2(g-1)}} \frac{S_{i_1 p}}{S_{0p}} \cdot \cdots \cdot \frac{S_{i_n p}}{S_{0p}}. \tag{B.2}$$

For $g = 1$ each summand is a product of all eigenvalues of the i_kth fusion matrix. Since the number of punctures and the weights i_k can be arbitrary and the left-hand side of the above equation is a natural number we conclude that the maximal eigenvalues have to be greater than 1. This concludes the proof of (B.1).

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