Quasi Riemann surfaces

Daniel Friedan

New High Energy Theory Center, Rutgers University
and Natural Science Institute, The University of Iceland
dfriedan@gmail.com

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Abstract

A *quasi Riemann surface* is defined to be a metric abelian group Q along with an epimorphism $Q \to \mathbb{Z}$ such that the integral currents in Q have properties analogous to the integral currents in a Riemann surface, sufficient for expressing Cauchy-Riemann equations on Q. The prototype is the metric abelian group $\mathcal{D}_0^{\mathrm{int}}(\Sigma)$ of integral 0-currents in a Riemann surface Σ. There is a bundle $Q(M)$ of quasi Riemann surfaces naturally associated to each oriented conformal $2n$-manifold M, formed from the bundle $\mathcal{D}_{n-1}^{\mathrm{int}}(M) \xrightarrow{\partial} \partial\mathcal{D}_{n-1}^{\mathrm{int}}(M)$ of integral $(n-1)$-currents in M fibered over the integral $(n-2)$-boundaries in M.

I suggest that complex analysis on quasi Rieman surfaces might be developed by analogy with classical complex analysis on Riemann surfaces. I hope that complex analysis on quasi Rieman surfaces can be used in constructing a new class of quantum field theories in spacetimes M as quantum field theories on the quasi Riemann surfaces $Q(M)$ by analogy with the construction of 2d conformal field theories on Riemann surfaces. The quasi Rieman surfaces $Q(M)$ might also be useful for studying the manifolds M.

Contents

1 Introduction

A quasi Riemann surface is defined to be a metric abelian group Q along with an epimorphism $Q \to \mathbb{Z}$ such that the integral currents in Q have properties analogous to the integral currents in a Riemann surface, sufficient for expressing Cauchy-Riemann equations on Q.

This note is based on an earlier paper [1] where the proposed definition of quasi Riemann surface was motivated by considerations from quantum field theory. Here the notion of quasi Riemann surface is presented without the quantum field theory motivation. The presentation is entirely naive and formal; there is no attempt at rigor; there is no attempt to be precise about topologies or domains of definition.

A summary of the quantum field theory project is given in [2]. The goal of the project is to construct quantum field theories on quasi Riemann surfaces by imitating the construction of 2d conformal field theories on Riemann surfaces. The latter is based on conformal tensor analysis on Riemann surfaces, especially the Laurent expansions of meromorphic conformal tensors and the Cauchy integral formula. An analogous technology is needed on quasi Riemann surfaces.

The basic objects are the *integral currents* in manifolds and in metric spaces provided by Geometric Measure Theory [3–5]. The metric abelian group $\mathcal{D}^{\mathrm{int}}_k(X)$ of integral k-currents in a manifold or metric space X is a certain metric completion of the abelian group of singular k-chains in X. Part of the initial impetus to consider this mathematical material came from Gromov's comments on spaces of cycles in section 5 of [6] which referred to [4].

The main elements of the proposal are:

1. For M an oriented $2n$-dimensional conformal manifold and Σ a Riemann surface, there is an analogy
$$\mathcal{D}^{\mathrm{int}}_{j+n-1}(M) \;\longleftrightarrow\; \mathcal{D}^{\mathrm{int}}_{j}(\Sigma) \qquad j = 0,1,2 \tag{1.1}$$
On each side there is a bilinear form, the *intersection form*, which gives the intersection number of two currents with $j_1 + j_2 = 2$, and there is a conformally invariant Hodge $*$-operator in the middle dimension, $j = 1$, acting on the vector space of differential forms dual to the currents.

2. The Cauchy-Riemann equations on a Riemann surface Σ can be expressed in terms of the integral currents in Σ, the boundary operator ∂, the intersection form, and the conformal Hodge $*$-operator acting in the middle dimension.

3. There are natural morphisms of metric abelian groups
$$\Pi_{j,k} \colon \mathcal{D}^{\mathrm{int}}_{j}(\mathcal{D}^{\mathrm{int}}_{k}(M)) \to \mathcal{D}^{\mathrm{int}}_{j+k}(M) \tag{1.2}$$
(Their construction is a basic point where mathematical rigor is wanted.)

4. $\mathcal{D}^{\mathrm{int}}_{n-1}(M)$ is regarded as a fiber bundle over the integral $(n-2)$-boundaries
$$\mathcal{D}^{\mathrm{int}}_{n-1}(M) \xrightarrow{\partial} \partial\mathcal{D}^{\mathrm{int}}_{n-1}(M) \qquad \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\partial\xi_0} = \left\{\xi \in \mathcal{D}^{\mathrm{int}}_{n-1}(M) \colon \partial\xi = \partial\xi_0\right\} \tag{1.3}$$
The examples $Q(M)$ of real quasi Riemann surfaces are the metric abelian groups
$$\text{for } n \text{ odd} \qquad Q_{\mathbb{Z}\partial\xi_0} = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} = \bigoplus_{m\in\mathbb{Z}} \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{m\partial\xi_0} \qquad Q_{\mathbb{Z}\partial\xi_0} \xrightarrow{\partial} \mathbb{Z}\partial\xi_0 \tag{1.4}$$

The examples of complex quasi Riemann surfaces are

$$\text{for } n \text{ even or odd} \qquad Q_{\mathbb{Z}\partial\xi_0} = \mathcal{D}^{\text{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}^{\text{int}}_n(M) \qquad Q_{\mathbb{Z}\partial\xi_0} \xrightarrow{\partial} \mathbb{Z}\partial\xi_0 \tag{1.5}$$

In both cases there is a chain complex of metric abelian groups

$$\begin{gathered} 0 \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} Q_2 \xrightarrow{\partial} Q_1 \xrightarrow{\partial} Q_0 \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} 0 \\ Q_0 = Q_{\mathbb{Z}\partial\xi_0} \qquad Q_1 = \mathcal{D}^{\text{int}}_n(M) \quad \text{or} \quad \mathcal{D}^{\text{int}}_n(M) \oplus i\mathcal{D}^{\text{int}}_n(M) \end{gathered} \tag{1.6}$$

with a nondegenerate intersection form and with an operator

$$J = \epsilon_n * \qquad \epsilon_n^2 = (-1)^{n-1} \qquad J^2 = -1 \tag{1.7}$$

acting on forms in the middle dimension, dual to Q_1. This is the same structure as found in the augmented chain complex of integral currents in a Riemann surface Σ

$$0 \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathcal{D}^{\text{int}}_2(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_1(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_0(\Sigma) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} 0 \tag{1.8}$$

The complexification (1.5) is needed when n is even because ϵ_n is then imaginary.

5. A version of tensor analysis on a metric abelian group Q is based on the integral currents $\mathcal{D}^{\text{int}}_j(Q)$. The space of j-forms on Q is defined as $\Omega_j(Q) = \text{Hom}(\mathcal{D}^{\text{int}}_j(Q), \mathbb{R})$. The tangent space $T_\xi Q$ at $\xi \in Q$ is defined as the vector space of infinitesimal 1-simplices at ξ. The cotangent space is the dual vector space $T^*_\xi Q = T_\xi Q^*$.

6. The intersection form and the conformal Hodge $*$-operator of M are pulled back to $Q = Q_{\mathbb{Z}\partial\xi_0}$ using morphisms $\Pi_j \colon \mathcal{D}^{\text{int}}_j(Q) \to Q_j$ derived from the morphisms

$$\Pi_{j,n-1} \colon \mathcal{D}^{\text{int}}_j(\mathcal{D}^{\text{int}}_{n-1}(M)) \to \mathcal{D}^{\text{int}}_{j+n-1}(M) \qquad j = 0,1,2 \tag{1.9}$$

The resulting structures on the $\mathcal{D}^{\text{int}}_j(Q)$, $j = 0,1,2$ are analogous to the structures on the integral currents $\mathcal{D}^{\text{int}}_j(\Sigma)$ in a Riemann surface Σ that are sufficient for expressing the Cauchy-Riemann equations.

7. A *quasi Riemann surface* is defined to be a metric abelian group Q, along with a morphism $Q \to \mathbb{Z}$, such that the integral currents $\mathcal{D}^{\text{int}}_j(Q)$ have these structures. The set of morphisms

$$Q_{\mathbb{Z}\partial\xi_0} \xrightarrow{\partial} \mathbb{Z}\partial\xi_0 \tag{1.10}$$

form a bundle of quasi Riemann surfaces $Q(M) \to B(M)$ naturally associated to the oriented conformal $2n$-manifold M. (More precisely, the base $B(M)$ is the space of maximal infinite cyclic subgroups $\mathbb{Z}\partial\xi_0 \subset \partial\mathcal{D}^{\text{int}}_{n-1}(M)$.)

8. A quantum field theory on M is to be constructed by putting a 2d conformal field theory on each of the quasi Riemann surfaces $Q_{\mathbb{Z}\partial\xi_0}$. There is to be one such quantum field theory on M for every ordinary 2d conformal field theory on Riemann surfaces.

This note consists of technical preliminaries (sections 2–5), the proposed version of tensor analysis on metric abelian groups (section 6), the examples $Q(M)$ (sections 7 and 8), and the proposed definition of quasi Riemann surface (section 9). A companion note [7] collects some questions, comments, and speculations and some remarks on complications that are glossed over here.

2 Integral currents in an oriented conformal $2n$-manifold M

2.1 The manifold M

Let M be an oriented conformal manifold of even dimension $2n \geq 2$. For simplicity, let M be compact and without boundary. The Hodge $*$-operator acting on smooth n-forms depends only on the conformal structure

$$
\begin{gathered}
*\colon \Omega_n^{\text{smooth}}(M) \to \Omega_n^{\text{smooth}}(M) \qquad * \, \omega_{\mu_1 \ldots \mu_n}(x) = \frac{1}{n!} \, \epsilon_{\mu_1 \ldots \mu_n}{}^{\nu_1 \ldots \nu_n}(x) \, \omega_{\nu_1 \ldots \nu_n}(x) \\
*^2 = (-1)^n
\end{gathered} \tag{2.1}
$$

The conformal Hodge $*$-operator acting on n-forms is all that we use of the conformal structure of M.

2.2 Currents and boundaries

A k-current ξ in M is a distribution on the smooth k-forms ω

$$
\xi\colon \omega \mapsto \int_\xi \omega = \int_M d^d x \, \frac{1}{k!} \xi^{\mu_1 \cdots \mu_k}(x) \, \omega_{\mu_1 \cdots \mu_k}(x) \qquad \deg(\xi) = k \tag{2.2}
$$

$\mathcal{D}_k^{\text{distr}}(M)$ is the real vector space of k-currents in M. The boundary operator on currents is the dual of the exterior derivative on forms

$$
\begin{gathered}
\partial\colon \mathcal{D}_k^{\text{distr}}(M) \to \mathcal{D}_{k-1}^{\text{distr}}(M) \qquad \int_{\partial\xi} \omega = \int_\xi d\omega \qquad (\partial\xi)^{\mu_2 \cdots \mu_k}(x) = -\partial_{\mu_1} \xi^{\mu_1 \cdots \mu_k}(x) \\
\partial^2 = 0
\end{gathered} \tag{2.3}
$$

The Hodge $*$-operator acts on the distributional n-currents by

$$
\int_{*\xi} \omega = \int_\xi *\omega \qquad \xi \in \mathcal{D}_n^{\text{distr}}(M) \qquad \omega \in \Omega_n^{\text{smooth}}(M) \tag{2.4}
$$

2.3 Singular currents

A k-simplex σ in M is represented by a k-current $[\sigma]$

$$
\sigma\colon \Delta^k \to M \qquad \int_{[\sigma]} \omega = \int_{\Delta^k} \sigma^* \omega \tag{2.5}
$$

The space $\mathcal{D}_k^{\text{sing}}(M)$ of *singular* k-currents in M is the abelian group of currents generated by the k-simplices in M, i.e., the currents representing the singular k-chains in M

$$
\sigma = \sum_i m_i \sigma_i \,, \quad m_i \in \mathbb{Z} \qquad [\sigma] = \sum_i m_i [\sigma_i] \qquad \int_{[\sigma]} \omega = \sum_i m_i \int_{\Delta^k} \sigma_i^* \omega \,. \tag{2.6}
$$

The boundary operators on singular chains and on singular currents are compatible

$$
\partial[\sigma] = [\partial\sigma] \,, \qquad \partial \mathcal{D}_k^{\text{sing}}(M) \subset \mathcal{D}_{k-1}^{\text{sing}}(M) \tag{2.7}
$$

2.4 Integral currents

The metric abelian group $\mathcal{D}_k^{\text{int}}(M)$ of *integral* k-currents in M is the metric completion of $\mathcal{D}_k^{\text{sing}}(M)$

$$\mathcal{D}_k^{\text{sing}}(M) \subset \mathcal{D}_k^{\text{int}}(M) \subset \mathcal{D}_k^{\text{distr}}(M) \tag{2.8}$$

with respect to the *flat metric* induced from the *flat norm*

$$\begin{gathered}\operatorname{dist}(\xi_1, \xi_2)_{\text{flat}} = \|\xi_1 - \xi_2\|_{\text{flat}} \\ \|\xi\|_{\text{flat}} = \inf \left\{ \operatorname{mass}(\xi - \partial\xi') + \operatorname{mass}(\xi'),\ \xi' \in \mathcal{D}_{k+1}^{\text{sing}}(M) \right\} \\ \operatorname{mass}(\xi) = k{-}\text{volume}(\xi) \qquad \xi \in \mathcal{D}_k^{\text{sing}}(M)\end{gathered} \tag{2.9}$$

The flat metric distance between ξ_1 and ξ_2 measures the effort needed to deform ξ_1 to ξ_2 or, equivalently, to deform $\xi_1 - \xi_2$ to 0. The k-volume of a k-current depends on a particular choice of Riemannian metric on M, but the resulting metric completion $\mathcal{D}_k^{\text{int}}(M)$ is independent of the choice. The boundary of an integral current is an integral current

$$\partial\mathcal{D}_k^{\text{int}}(M) \subset \mathcal{D}_{k-1}^{\text{int}}(M) \tag{2.10}$$

2.5 The fiber bundle of integral k-currents

Regard

$$\mathcal{D}_k^{\text{int}}(M) \xrightarrow{\partial} \partial\mathcal{D}_k^{\text{int}}(M) \tag{2.11}$$

as a fiber bundle with fibers

$$\mathcal{D}_k^{\text{int}}(M)_{\partial\xi_0} = \left\{ \xi \in \mathcal{D}_k^{\text{int}}(M) \colon \partial\xi = \partial\xi_0 \right\} \tag{2.12}$$

$\mathcal{D}_k^{\text{int}}(M)_0 = \operatorname{Ker}\partial$ is the metric abelian group of integral k-cycles. For $\partial\xi_0 \neq 0$, the fiber $\mathcal{D}_k^{\text{int}}(M)_{\partial\xi_0}$ is the space of *relative* integral k-cycles, i.e., relative to $\partial\xi_0$.

2.6 The intersection form

The bilinear *intersection form* is defined almost everywhere on pairs of currents in M, vanishing unless the degrees of the two currents add up to the dimension of M

$$\begin{aligned}&I_M(\xi_1, \xi_2) = \int_M d^d x\, \frac{1}{k_1! k_2!} \epsilon_{\mu_1 \cdots \mu_{k_1} \nu_1 \cdots \nu_{k_2}}(x)\, \xi_1^{\mu_1 \cdots \mu_{k_1}}(x)\, \xi_2^{\nu_1 \cdots \nu_{k_2}}(x) \qquad k_1 + k_2 = 2n \\ &I_M(\xi_1, \xi_2) = 0 \qquad k_1 + k_2 \neq 2n\end{aligned} \tag{2.13}$$

The intersection form is independent of the conformal structure; it depends only on the orientation of M. The intersection form on integral currents gives the integer intersection number

$$I_M(\xi_1, \xi_2) \in \mathbb{Z} \quad \text{(where defined)} \qquad \xi_1 \in \mathcal{D}_{k_1}^{\text{int}}(M) \qquad \xi_2 \in \mathcal{D}_{k_2}^{\text{int}}(M) \tag{2.14}$$

The intersection form on the integral cycles is defined everywhere and depends only on the homology classes of the cycles.

2.7 The chain complex of integral currents

For $n \geq 2$ we use only a portion of the middle of the chain complex of integral currents

$$\mathcal{D}^{\text{int}}_{n+2}(M) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{n+1}(M) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{n}(M) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{n-1}(M) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{n-2}(M) \tag{2.15}$$

When $n = 1$, M is a two-dimensional conformal manifold, i.e., a Riemann surface, which we write Σ instead of M. We augment its integral chain complex at both ends

$$0 \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{3}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{2}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{1}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{0}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{-1}(\Sigma) \xrightarrow{\partial} 0 \tag{2.16}$$

where we set

$$\begin{aligned} &\mathcal{D}^{\text{int}}_{-1}(\Sigma) = \mathbb{Z} \qquad \eta \in \mathcal{D}^{\text{int}}_{0}(\Sigma) \overset{\partial}{\longmapsto} \int_\eta 1 \in \mathcal{D}^{\text{int}}_{-1}(\Sigma) \\ &\mathcal{D}^{\text{int}}_{3}(\Sigma) = \mathbb{Z} \qquad 1 \in \mathcal{D}^{\text{int}}_{3}(\Sigma) \overset{\partial}{\longmapsto} \Sigma \in \mathcal{D}^{\text{int}}_{2}(\Sigma) \end{aligned} \tag{2.17}$$

In particular

$$\partial \delta_z = 1 \tag{2.18}$$

where $\delta_z \in \mathcal{D}^{\text{int}}_{0}(\Sigma)$ represents the point $z \in \Sigma$, i.e., δ_z is the Dirac delta-function at z.

In order to maintain a uniform terminology over all n, we call $\mathcal{D}^{\text{int}}_{0}(\Sigma)$ the space of integral 0-currents and we call $\operatorname{Ker} \partial = \mathcal{D}^{\text{int}}_{0}(\Sigma)_0$ the space of integral 0-cycles. It is perhaps more usual to write the chain complex without augmentation

$$0 \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{2}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{1}(\Sigma) \xrightarrow{\partial} \mathcal{D}^{\text{int}}_{0}(\Sigma) \xrightarrow{\partial} 0 \tag{2.19}$$

Then $\mathcal{D}^{\text{int}}_{0}(\Sigma)$ is called the space of integral 0-cycles and, for $\eta \in \mathcal{D}^{\text{int}}_{0}(\Sigma)$, the integer $\partial \eta$ defined in (2.17) is called the degree of η.

3 Integral currents in spaces of integral currents

The integral currents in a complete metric space were constructed in [5]. So we take as given the metric abelian group $\mathcal{D}^{\text{int}}_{j}(\mathcal{D}^{\text{int}}_{k}(M))$ of integral j-currents in the complete metric space $\mathcal{D}^{\text{int}}_{k}(M)$.

3.1 The morphisms $\Pi_{j,k} \colon \mathcal{D}^{\text{int}}_{j}(\mathcal{D}^{\text{int}}_{k}(M)) \to \mathcal{D}^{\text{int}}_{j+k}(M)$

Following comments in section 5 of [6] which refer to [4], we write $\mathcal{C}^{\text{sing}}_{k}(M)$ for the space of singular k-chains in M and $\mathcal{C}^{\text{sing}}_{j}(\mathcal{C}^{\text{sing}}_{k}(M))$ for the space of singular j-chains in $\mathcal{C}^{\text{sing}}_{k}(M)$. The product $\Delta^j \times \Delta^k$ of the j-simplex with the k-simplex is a singular $(j+k)$-chain so there is a natural morphism

$$\Pi^{\text{sing}}_{j,k} \colon \mathcal{C}^{\text{sing}}_{j}(\mathcal{C}^{\text{sing}}_{k}(M)) \to \mathcal{C}^{\text{sing}}_{j+k}(M) \tag{3.1}$$

We suppose that $\Pi^{\text{sing}}_{j,k}$ respects the flat metrics so gives a natural morphism of metric abelian groups

$$\Pi_{j,k} \colon \mathcal{D}^{\text{int}}_{j}(\mathcal{D}^{\text{int}}_{k}(M)) \to \mathcal{D}^{\text{int}}_{j+k}(M) \tag{3.2}$$

In particular

$$\Pi_{0,k} \delta_\xi = \xi \tag{3.3}$$

where $\delta_\xi \in \mathcal{D}^{\text{int}}_{0}(\mathcal{D}^{\text{int}}_{k}(M))$ is the 0-current representing the point $\xi \in \mathcal{D}^{\text{int}}_{k}(M)$.

3.2 $\Pi_{j,k}$ and ∂

From

$$\partial(\Delta^j \times \Delta^k) = \partial\Delta^j \times \Delta^k + (-1)^j \Delta^j \times \partial\Delta^k \tag{3.4}$$

it follows that

$$\partial\Pi_{j,k} = \Pi_{j-1,k}\partial + (-1)^j \Pi_{j,k-1}\partial_{*j,k} \tag{3.5}$$

where

$$\partial_{*j,k} \colon \mathcal{D}_j^{\mathrm{int}}(\mathcal{D}_k^{\mathrm{int}}(M)) \to \mathcal{D}_j^{\mathrm{int}}(\mathcal{D}_{k-1}^{\mathrm{int}}(M)) \tag{3.6}$$

is the push-forward of the boundary map $\partial \colon \mathcal{D}_k^{\mathrm{int}}(M) \to \mathcal{D}_{k-1}^{\mathrm{int}}(M)$.

3.3 Translation invariance

Let T^ξ be translation by ξ in the abelian group $\mathcal{D}_k^{\mathrm{int}}(M)$

$$\mathrm{T}^\xi \colon \mathcal{D}_k^{\mathrm{int}}(M) \to \mathcal{D}_k^{\mathrm{int}}(M) \qquad \mathrm{T}^\xi \colon \xi' \mapsto \xi + \xi' \tag{3.7}$$

T^ξ acts on currents in $\mathcal{D}_k^{\mathrm{int}}(M)$ by pushing forward

$$\mathrm{T}^\xi_* \colon \mathcal{D}_j^{\mathrm{int}}(\mathcal{D}_k^{\mathrm{int}}(M)) \to \mathcal{D}_j^{\mathrm{int}}(\mathcal{D}_k^{\mathrm{int}}(M)) \tag{3.8}$$

From $\Pi_{0,k}\delta_\xi = \xi$ it follows that $\Pi_{0,k}$ is translation-invariant in the sense that

$$\Pi_{0,k}\mathrm{T}^\xi_* = \mathrm{T}^\xi \Pi_{0,k} \tag{3.9}$$

The $\Pi_{j,k}$ for $j \geq 1$ are translation-invariant in the sense that

$$\Pi_{j,k}\mathrm{T}^\xi_* = \Pi_{j,k} \qquad j \geq 1 \tag{3.10}$$

This follows from the fact that a map from $\Delta^j \times \Delta^k$ to M which is constant on Δ^k is represented by 0 as a $(j+k)$-current in M if $j \geq 1$.

4 Modify conformal Hodge-$*$ and the intersection form to be independent of n

Some trivial modifications can be made to the conformal Hodge $*$-operator and to the intersection form $I_M(\xi_1, \xi_2)$ so that their properties on $(j+n-1)$-currents become the same for all n. When pulled back to $\mathcal{D}_j^{\mathrm{int}}(\mathcal{D}_{n-1}^{\mathrm{int}}(M))$ along $\Pi_{j,n-1}$ their properties on j-currents in $\mathcal{D}_{n-1}^{\mathrm{int}}(M)$ will then be the same for all n. To accomplish this for both even and odd values of n requires using the complex currents

$$\mathcal{D}_k^{\mathrm{distr}}(M, \mathbb{C}) = \mathcal{D}_k^{\mathrm{distr}}(M) \otimes \mathbb{C} \tag{4.1}$$

Choose a root ϵ_n of the equation

$$\epsilon_n^2 = (-1)^{n-1} \qquad \epsilon_1 = 1 \tag{4.2}$$

then define

$$\deg'(\xi) = \deg(\xi) - (n-1) = k - n + 1 \qquad \xi \in \mathcal{D}_k^{\mathrm{distr}}(M, \mathbb{C}) \tag{4.3}$$

$$J = \epsilon_n * \quad \text{acting on } \mathcal{D}_n^{\mathrm{distr}}(M, \mathbb{C}) \tag{4.4}$$

$$I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle = \epsilon_n^{-1}\, (-1)^{\frac{1}{2}(\deg'(\xi_2)-1)(\deg'(\xi_2)+2n)}\, I_M(\bar{\xi}_1, \xi_2) \tag{4.5}$$

where $\bar{\xi}$ is the complex conjugate of the current ξ. These satisfy (wherever defined) a set of properties that make no mention of n

$$I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle \neq 0 \quad \text{only if } \deg'(\xi_1) + \deg'(\xi_2) = 2 \tag{4.6}$$

$$I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle = -\overline{I_M\langle\, \bar{\xi}_2, \xi_1 \,\rangle} \tag{4.7}$$

$$I_M\langle\, \overline{\partial\xi_1}, \xi_2 \,\rangle = -I_M\langle\, \bar{\xi}_1, \partial\xi_2 \,\rangle \tag{4.8}$$

$$J \text{ acts on the } \deg'(\xi) = 1 \text{ subspace} \tag{4.9}$$

$$J^2 = -1 \tag{4.10}$$

$$I_M\langle\, \overline{J\xi_1}, J\xi_2 \,\rangle = I_M\langle\, \bar{\xi}_1, \xi_2 \,\rangle \qquad \deg'(\xi_1) = \deg'(\xi_2) = 1 \tag{4.11}$$

$$I_M\langle\, \bar{\xi}, J\xi \,\rangle > 0 \qquad \deg'(\xi) = 1,\ \xi \neq 0 \tag{4.12}$$

We call $I_M\langle\, \bar{\xi}_2, \xi_1 \,\rangle$ the *skew-hermitian intersection form* because of property (4.7). The number ϵ_n is real when n is odd so we can restrict to real currents. Then we call $I_M\langle\, \xi_1, \xi_2 \,\rangle$ the *skew intersection form.*

When $n = 1$, when M is a Riemann surface Σ, the skew intersection form extends uniquely to $\mathcal{D}_3^{\mathrm{int}}(\Sigma) \times \mathcal{D}_{-1}^{\mathrm{int}}(\Sigma) = \mathbb{Z} \times \mathbb{Z}$ by

$$I_\Sigma\langle\, 1, 1 \,\rangle = I_\Sigma\langle\, 1, \partial\delta_z \,\rangle = -I_\Sigma\langle\, \partial 1, \delta_z \,\rangle = -I_\Sigma\langle\, \Sigma, \delta_z \,\rangle = I_\Sigma\langle\, \delta_z, \Sigma \,\rangle = \epsilon_1^{-1} I_\Sigma(\delta_z, \Sigma) = 1 \tag{4.13}$$

5 Cauchy-Riemann equations

Let Σ be a Riemann surface with local real coordinates $x = (x^1, x^2)$ and local complex coordinate $z = x^1 + ix^2$. $J = *$ acts on 1-forms by

$$J dz = i dz \qquad J d\bar{z} = -i d\bar{z} \tag{5.1}$$

A fundamental solution of the Cauchy-Riemann equations

$$F(x_0, x)_\mu dx^\mu = F(z_0, z) dz = \frac{dz}{z - z_0} + \cdots \tag{5.2}$$

is a 1-form in z and a function of z_0 which satisfies

$$F(J - i) = 0 \qquad \frac{\partial}{\partial \bar{z}} F(z_0, z) = \pi \delta^2(z - z_0) \tag{5.3}$$

Regard F as a complex form (defined almost everywhere)

$$F\colon\ \overline{\mathcal{D}_0^{\rm int}(\Sigma)}\times\mathcal{D}_1^{\rm int}(\Sigma)\to\mathbb{C} \tag{5.4}$$

$$F(\bar\xi_0,\xi_1)=\int_\Sigma d^2x_0\int_\Sigma d^2x\ \overline{\xi_0(x_0)}\ F(x_0;x)_\mu\,\xi_1^\mu(x) \tag{5.5}$$

(The integral currents in Σ are real so complex conjugation acts trivially and is only written for the sake of later generalization.) Write $F(\bar\xi_0,\xi_1)$ also for the linear extension to a form on the complex currents (defined almost everywhere)

$$F\colon\ \overline{\mathcal{D}_0^{\rm distr}(\Sigma,\mathbb{C})}\otimes\mathcal{D}_1^{\rm distr}(\Sigma,\mathbb{C})\to\mathbb{C}\qquad \mathcal{D}_j^{\rm distr}(\Sigma,\mathbb{C})=\mathcal{D}_j^{\rm distr}(\Sigma)\otimes\mathbb{C} \tag{5.6}$$

The Cauchy-Riemann equations (5.3) become

$$F(\bar\xi_0,\partial\xi_2)=2\pi i I_\Sigma\langle\,\bar\xi_0,\xi_2\,\rangle\qquad \xi_0\in\mathcal{D}_0^{\rm int}(\Sigma)\qquad\quad \xi_2\in\mathcal{D}_2^{\rm int}(\Sigma) \tag{5.7}$$

$$F(\bar\xi_0,(J-i)\xi_1)=0\qquad\qquad \xi_0\in\mathcal{D}_0^{\rm distr}(\Sigma,\mathbb{C})\qquad \xi_1\in\mathcal{D}_1^{\rm distr}(\Sigma,\mathbb{C}) \tag{5.8}$$

The Cauchy-Riemann equations are thus expressed in terms of the integral currents, the skew-hermitian intersection form, and the J operator acting on complex currents.

(The Cauchy-Riemann equations should be written locally on neighborhoods $U\subset\Sigma$, substituting U for Σ in the above.)

6 Construct forms and tangent vectors from the integral currents

The aim is to develop conformal tensor analysis on a metric abelian group Q such as the $Q_{\mathbb{Z}\partial\xi_0}$ of equations (1.4) and (1.5), in analogy with conformal tensor analysis on Riemann surfaces. We want to lift the skew-hermitian intersection form $I_M\langle\,\bar\xi_1,\xi_2\,\rangle$ and the operator J from the manifold M to the metric space $Q=Q_{\mathbb{Z}\partial\xi_0}$ via the maps

$$\Pi_{j,n-1}\colon\mathcal{D}_j^{\rm int}(Q)\to\mathcal{D}_{j+n-1}^{\rm int}(M) \tag{6.1}$$

To do this we need a construction of forms on Q. But there is no smooth manifold structure available on Q in which to construct forms. What *is* available are the currents in Q constructed in [5]. The j-currents in a complete metric space such as Q are constructed in [5] not as distributions on j-forms but rather as linear functionals on $(j{+}1)$-tuplets of Lipschitz functions. Only the metric structure of Q is used; there is no need of a smooth structure. We want to pull back forms from M to Q via the maps $\Pi_{j,n-1}$ between spaces of integral currents, so we construct the forms from the *integral* currents. We take the integral currents $\mathcal{D}_j^{\rm int}(Q)$ as the foundation for tensor analysis on Q. Essential use is made of the abelian group structure of Q. This construction of tensor analysis works as well for any space that looks locally like a metric abelian group. For a manifold M it reproduces the usual forms and the usual tensor analysis.

6.1 Forms on Q

Define a j-form on Q to be a homomorphism of abelian groups

$$\omega\colon \mathcal{D}_j^{\mathrm{int}}(Q) \to \mathbb{R} \qquad \omega(\eta_1+\eta_2) = \omega(\eta_1)+\omega(\eta_2) \tag{6.2}$$

satisfying some topological or metric regularity condition. The regularity condition will presumably depend on the application. Here we proceed formally, hoping that the formal structure will in the end provide criteria for precise definitions.

$\mathcal{D}_j^{\mathrm{int}}(Q)$ is generated as a metric abelian group by the arbitrarily small j-simplices in Q, so a j-form on Q is determined by its values on the infinitesimal j-simplices in Q, i.e., by local data on Q. The space of j-forms

$$\Omega_j(Q) = \mathrm{Hom}(\mathcal{D}_j^{\mathrm{int}}(Q), \mathbb{R}) \tag{6.3}$$

is a real vector space. The exterior derivative is the dual of the boundary operator

$$d\colon \Omega_j(Q) \to \Omega_{j+1}(Q) \qquad d\omega(\xi) = \omega(\partial\xi) \qquad d^2 = 0 \tag{6.4}$$

6.2 Currents in Q

Define the j-currents in Q to be the real linear functions on j-forms

$$\mathcal{D}_j(Q) = \Omega_j(Q)^* \qquad \xi\colon \omega \mapsto (\omega,\xi) \qquad \partial\colon \mathcal{D}_j(Q) \to \mathcal{D}_{j-1}(Q) \qquad \partial = d^* \tag{6.5}$$

Again this is a formal definition. We expect $\mathcal{D}_j(Q)$ to be a linear subspace of the distributional j-currents in Q constructed in [5]

$$\mathcal{D}_j(Q) \subset \mathcal{D}_j^{\mathrm{distr}}(Q) \tag{6.6}$$

$\mathcal{D}_j^{\mathrm{int}}(Q)$ is generated by the arbitrarily small integral j-currents so we can identify $\mathcal{D}_j(Q)$ with the space of infinitesimal integral j-currents, something akin to or even equal to the Gromov-Hausdorff tangent space

$$\mathcal{D}_j(Q) = T_0^{\mathrm{G\text{-}H}}(\mathcal{D}_j^{\mathrm{int}}(Q)) \tag{6.7}$$

6.3 The tangent and cotangent bundles of Q

The infinitesimal j-simplices at a point $\xi \in Q$ form a vector space $T_{j,\xi}Q$ because Q is an abelian group. Call this the space of tangent j-vectors at ξ. The vector spaces $T_{j,\xi}Q$ are the fibers of a vector bundle $T_jQ \to Q$. The dual vector spaces $T^*_{j,\xi}Q = T_{j,\xi}Q^*$ are the fibers of a vector bundle $T^*_jQ \to Q$. The tangent bundle is $TQ = T_1Q$. The cotangent bundle is $T^*Q = T^*_1Q$. All of these vector bundles are product bundles because Q is an abelian group

$$T_{j,\xi}Q = T_{j,0}Q \qquad T_jQ = Q \times T_{j,0}Q \qquad T^*_{j,\xi}Q = T^*_{j,0}Q \qquad T^*_jQ = Q \times T^*_{j,0}Q \tag{6.8}$$

The j-forms on Q are the sections of T^*_jQ

$$\Omega_j(Q) = \Gamma(T^*_jQ) \tag{6.9}$$

The rest of this subsection goes into these points in a bit more detail.

Let $t = (t_1, \ldots, t_j)$ be coordinates for $\mathbb{R}^j$. Let R_a be the dilation

$$R_a(t) = (at_1, at_2, \ldots, at_j) \qquad a > 0 \tag{6.10}$$

For $j = 0$ define

$$T_{0,\xi}Q = \mathbb{R}\delta_\xi \subset \mathcal{D}_0(Q) \tag{6.11}$$

For $j \geq 1$ parametrize the j-simplex Δ^j as the j-cube $[0,1]^j \subset \mathbb{R}^j$. Define a j-simplex *at* ξ to be a map $\sigma\colon [0,1]^j \to Q$ with $\sigma(0) = \xi$. The *tangent j-vector* to σ at $t = 0$ is the j-current $D\sigma(0)$ given by

$$(\omega, D\sigma(0)) = \lim_{\epsilon \downarrow 0} \epsilon^{-j}\omega([\sigma R_\epsilon]) \qquad \omega \in \Omega_j(Q) = \mathrm{Hom}(\mathcal{D}_j^{\mathrm{int}}(Q), \mathbb{R}) \tag{6.12}$$

$$D\sigma(0) \in \mathcal{D}_j(Q) = \Omega_j(Q)^* \tag{6.13}$$

$D\sigma(0)$ should be the same as the distributional j-current

$$D\sigma(0) = \lim_{\epsilon \downarrow 0} \epsilon^{-j}[\sigma R_\epsilon] \in \mathcal{D}_j^{\mathrm{distr}}(Q) \tag{6.14}$$

Define the space of tangent j-vectors at ξ to be

$$T_{j,\xi}Q = \{D\sigma(0)\colon\ \sigma \text{ a } j\text{-simplex at } \xi\} \subset \mathcal{D}_j(Q) \tag{6.15}$$

$T_{j,\xi}Q$ is a vector space because

$$D(\sigma R_a)(0) = a^j D\sigma(0) \qquad D(\sigma_1 + \sigma_2)(0) = D\sigma_1(0) + D\sigma_2(0) \tag{6.16}$$

where the abelian group structure of Q is used to add j-simplices

$$(\sigma_1 + \sigma_2)(t) = \sigma_1(t) + \sigma_2(t) \tag{6.17}$$

$T_{j,\xi}Q$ might be characterized as the distributional j-currents in Q supported strictly on ξ.

Define the space of cotangent j-vectors at ξ to be the dual vector space

$$T^*_{j,\xi}Q = T_{j,\xi}Q^* \tag{6.18}$$

which is the space of equivalence classes

$$T^*_{j,\xi}Q = \Omega_j(Q)/N_{j,\xi}(Q) \tag{6.19}$$

of j-forms modulo the null subspace under the pairing with the tangent j-vectors at ξ

$$N_{j,\xi}(Q) = \{\omega \in \Omega_j(Q)\colon (\omega, D\sigma(0)) = 0\,,\ \forall\, D\sigma(0) \in T_{j,\xi}Q\} \tag{6.20}$$

The projections on the quotients are the evaluation maps

$$\Omega_j(Q) \to T^*_{j,\xi}Q \qquad \omega \mapsto \omega(\xi) \tag{6.21}$$

representing the j-forms as the sections of the bundle of cotangent j-vectors

$$\Omega_j(Q) = \Gamma(T^*_j Q) \tag{6.22}$$

in the appropriate sense of section.

The translation operators T^ξ identify all the j-tangent vector spaces and all the j-cotangent vector spaces

$$T_{j,\xi}Q = T^\xi_*(T_{j,0}Q) \qquad T^*_{j,\xi}Q = T^{\xi *}(T^*_{j,0}Q) \tag{6.23}$$

so the vector bundles can be written as product bundles

$$T_jQ = Q \times T_{j,0}Q \qquad T^*_jQ = Q \times T^*_{j,0}Q \tag{6.24}$$

The j-cotangent space can be identified with the translation-invariant j-forms

$$T^*_{j,0}Q = \Omega_j(Q)_{inv} = \left\{\omega \in \Omega_j(Q) \colon T^{\xi *}\omega = \omega \;\; \forall \xi \in Q\right\} \tag{6.25}$$

6.4 $d\Pi_{j,n-1} \colon T_jQ \to \mathcal{D}_{j+n-1}(M)$

The morphism

$$\Pi_{j,n-1} \colon \mathcal{D}^{\text{int}}_j(Q) \to \mathcal{D}^{\text{int}}_{j+n-1}(M) \tag{6.26}$$

acts on the infinitesimal j-simplices in Q as a translation-invariant linear map

$$d\Pi_{j,n-1} \colon T_jQ \to \mathcal{D}_{j+n-1}(M) \tag{6.27}$$

$d\Pi_{1,n-1}$ identifies each tangent space $T_\xi Q$ with the linear space $\mathcal{D}_n(M)$

$$d\Pi_{1,n-1} \colon T_\xi Q \xrightarrow{\sim} \mathcal{D}_n(M) \subset \mathcal{D}^{\text{distr}}_n(M) \tag{6.28}$$

where $\mathcal{D}_n(M)$ is the space of infinitesimal integral n-currents in M as in section 6.2 above. $\mathcal{D}_n(M)$ is a dense linear subspace of $\mathcal{D}^{\text{distr}}_n(M)$ consisting (at least roughly) of the n-currents in M that are strictly supported on an integral $(n{-}1)$-current. The crucial point is that $\mathcal{D}_n(M)$ is closed under the conformal Hodge $*$-operator

$$*\mathcal{D}_n(M) = \mathcal{D}_n(M) \tag{6.29}$$

The germ of a proof of this is given in Appendix 1 of [1]. The argument makes essential use of fractal integral n-currents which are the limits of Cauchy sequences of singular n-currents with respect to the flat metric. This is one of the two main motivations for using the integral currents. The other is the (presumed) existence of the maps $\Pi_{j,k}$. Assuming (6.29), the conformal Hodge $*$-operator acts on the tangent spaces of Q

$$* \colon T_\xi Q \to T_\xi Q \tag{6.30}$$

7 The real examples $Q(M)$ for n odd

Assume n odd. Let Q be one of the metric abelian groups

$$Q = \mathcal{D}^{\text{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} = \big\{\xi \in \mathcal{D}^{\text{int}}_{n-1}(M) \colon \partial\xi \in \mathbb{Z}\partial\xi_0\big\} \qquad \partial\xi_0 \in \partial\mathcal{D}^{\text{int}}_{n-1}(M) \tag{7.1}$$

with $\mathbb{Z}\partial\xi_0$ a *maximal* cyclic subgroup of $\partial\mathcal{D}^{\text{int}}_{n-1}(M)$. That is, $\partial\xi_0 = N\partial\xi_0'$, $N \in \mathbb{Z}$, only if $N = \pm 1$.

7.1 A chain complex $\oplus_j Q_j$ analogous to $\oplus_j \mathcal{D}^{\text{int}}_j(\Sigma)$

For each Q construct a chain complex of metric abelian groups analogous to the augmented chain complex (2.16) of integral currents $\mathcal{D}^{\text{int}}_j(\Sigma)$ in a Riemann surface Σ

$$0 \xrightarrow{\partial} Q_3 \xrightarrow{\partial} Q_2 \xrightarrow{\partial} Q_1 \xrightarrow{\partial} Q_0 \xrightarrow{\partial} Q_{-1} \xrightarrow{\partial} 0 \tag{7.2}$$

$$\begin{gathered} Q_0 = Q \subset \mathcal{D}^{\text{int}}_{n-1}(M) \qquad Q_1 = \mathcal{D}^{\text{int}}_{n}(M) \qquad Q_2 = \mathcal{D}^{\text{int}}_{n+1}(M)/Q_0^{\perp} \\ Q_{-1} = \mathbb{Z}\partial\xi_0 \subset \mathcal{D}^{\text{int}}_{n-2}(M) \qquad Q_3 = \mathcal{D}^{\text{int}}_{n+2}(M)/Q_{-1}^{\perp} \qquad Q_{-1} \cong \mathbb{Z} \qquad Q_3 \cong \mathbb{Z} \end{gathered} \tag{7.3}$$

where $Q_0^{\perp}$ and $Q_{-1}^{\perp}$ are the orthogonal complements in the skew intersection form of M

$$Q_0^{\perp} = \big\{\xi \in \mathcal{D}^{\text{int}}_{n+1}(M) :\ I_M\langle\, \xi', \xi\,\rangle = 0 \quad \forall \xi' \in Q_0\big\} \tag{7.4}$$

$$Q_{-1}^{\perp} = \big\{\xi \in \mathcal{D}^{\text{int}}_{n+2}(M) :\ I_M\langle\, \xi', \xi\,\rangle = 0 \quad \forall \xi' \in Q_{-1}\big\} \tag{7.5}$$

The Q_j form a chain complex because $Q_{-1} = \partial Q_0$ so $\partial Q_{-1}^{\perp} = Q_0^{\perp}$ by the integration by parts property (4.8) of the skew intersection form $I_M\langle\, \xi_1, \xi_2\,\rangle$.

7.2 The skew form $I\langle\, \xi_1, \xi_2\,\rangle$ on $\oplus_j Q_j$

By construction, $I_M\langle\, \xi_1, \xi_2\,\rangle$ defines (almost everywhere) an integer-valued form

$$I\langle\, \xi_1, \xi_2\,\rangle \in \operatorname{Hom}(\oplus_j Q_j \times \oplus_j Q_j, \mathbb{Z}) \tag{7.6}$$

satisfying (where defined)

$$I\langle\, \xi_1, \xi_2\,\rangle = 0 \qquad \begin{aligned} &\text{unless } \deg'(\xi_1) + \deg'(\xi_2) = 2 \\ &\text{where } \deg'(\xi) = j \text{ for } \xi \in Q_j \end{aligned} \tag{7.7}$$

$$I\langle\, \xi_1, \xi_2\,\rangle = -I\langle\, \xi_2, \xi_1\,\rangle \tag{7.8}$$

$$I\langle\, \partial\xi_1, \xi_2\,\rangle = -I\langle\, \xi_1, \partial\xi_2\,\rangle \tag{7.9}$$

$$I\langle\, \xi_1, \xi_2\,\rangle \text{ is nondegenerate} \tag{7.10}$$

7.3 The J operator

Define real vector spaces

$$\Omega_j = \mathrm{Hom}(Q_j, \mathbb{R}) \qquad \mathcal{D}_j = \Omega_j^* \tag{7.11}$$

analogous to the vector spaces $\Omega_j(\Sigma)$ and $\mathcal{D}_j(\Sigma)$ of forms and currents on a Riemann surface constructed from its integral currents as in section 6 above. In particular

$$\Omega_1 = \mathrm{Hom}(\mathcal{D}_n^{\mathrm{int}}(M), \mathbb{R}) \qquad \mathcal{D}_1 = \mathcal{D}_n(M) \subset \mathcal{D}_n^{\mathrm{distr}}(M) \tag{7.12}$$

Recall the definition $J = \epsilon_n *$ with $\epsilon_n^2 = (-1)^{n-1}$ and the assumption that $*\mathcal{D}_n(M) = \mathcal{D}_n(M)$. Now we have the final elements of the analogy:

$$J^2 = -1 \quad \text{on } \mathcal{D}_1 \tag{7.13}$$

$$I\langle\, J\xi_1, J\xi_2 \,\rangle = I\langle\, \xi_1, \xi_2 \,\rangle \qquad \xi_1, \xi_2 \in \mathcal{D}_1 \tag{7.14}$$

$$I\langle\, \xi, J\xi \,\rangle > 0 \qquad \xi \neq 0 \in \mathcal{D}_1 \tag{7.15}$$

where $I\langle\, \xi_1, \xi_2 \,\rangle$ is extended from Q_1 to $\mathcal{D}_1$ by linearity. Note that the positivity condition (7.15) cannot be literally true. The vector space $\mathcal{D}_1 = \mathcal{D}_n(M)$ needs to be extended in some natural way to become large enough to include a dense set of L_2 vectors.

7.4 A morphism of chain complexes $\Pi\colon \oplus_j \mathcal{D}_j^{\mathrm{int}}(Q) \to \oplus_j Q_j$

There is a morphism of chain complexes of metric abelian groups,

$$\cdots \begin{array}{ccccccccccccc} \mathcal{D}_4^{\mathrm{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_3^{\mathrm{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_2^{\mathrm{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_1^{\mathrm{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_0^{\mathrm{int}}(Q) & \xrightarrow{\partial} & \mathcal{D}_{-1}^{\mathrm{int}}(Q) & \xrightarrow{\partial} & 0 \\ \downarrow{\scriptstyle \Pi_4} & & \downarrow{\scriptstyle \Pi_3} & & \downarrow{\scriptstyle \Pi_2} & & \downarrow{\scriptstyle \Pi_1} & & \downarrow{\scriptstyle \Pi_0} & & \downarrow{\scriptstyle \Pi_{-1}} & & \\ 0 & \xrightarrow{\partial} & Q_3 & \xrightarrow{\partial} & Q_2 & \xrightarrow{\partial} & Q_1 & \xrightarrow{\partial} & Q_0 & \xrightarrow{\partial} & Q_{-1} & \xrightarrow{\partial} & 0 \end{array} \tag{7.16}$$

where the top complex is augmented on the right by

$$\mathcal{D}_{-1}^{\mathrm{int}}(Q) = Q_{-1} = \mathbb{Z}\partial\xi_0 \qquad \partial\colon \delta_\xi \in \mathcal{D}_0^{\mathrm{int}}(Q) \mapsto \partial\xi \in \mathbb{Z}\partial\xi_0 \tag{7.17}$$

The morphism maps are

$$\Pi_j = 0 \quad j \geq 4 \qquad \Pi_j = \begin{cases} \Pi_{j,n-1} & j = 0,1 \\ \pi_j \circ \Pi_{j,n-1} & j = 2,3 \end{cases} \qquad \Pi_{-1} = 1 \tag{7.18}$$

where π_j is the projection on the quotient space

$$\pi_j\colon \mathcal{D}_{j+n-1}^{\mathrm{int}}(M) \to Q_j = \mathcal{D}_{j+n-1}^{\mathrm{int}}(M)/Q_{2-j}^{\perp} \qquad j = 2,3 \tag{7.19}$$

In particular

$$\Pi_0\colon \mathcal{D}_0^{\mathrm{int}}(Q) \to Q \qquad \delta_\xi \mapsto \xi \tag{7.20}$$

To see that Π is a morphism of chain complexes first verify explicitly

$$\partial \Pi_0 = \Pi_{-1} \partial \tag{7.21}$$

Then note that

$$\partial \Pi_j = \Pi_{j-1} \partial \qquad j \geq 1 \tag{7.22}$$

because the operator $\partial_{*j,n-1}$ in equation (3.5) vanishes on $\mathcal{D}^{\text{int}}_j(Q)$ for $j \geq 1$ because ∂ takes Q to the discrete set $\mathbb{Z}\partial\xi_0$ and there are no nonzero j-currents in a discrete set if $j \geq 1$.

7.5 Translation invariance of Π

Translations in Q act as automorphisms of the two chain complexes

$$\mathrm{T}^\xi_* \colon \mathcal{D}^{\text{int}}_j(Q) \to \mathcal{D}^{\text{int}}_j(Q) \qquad \mathrm{T}^\xi \colon Q_j \to Q_j \qquad \xi \in Q \tag{7.23}$$

where T^ξ acts as translation on $Q_0 = Q$ and acts trivially on the Q_j for $j \geq 1$. On $\mathcal{D}^{\text{int}}_{-1}(Q) = Q_{-1}$

$$T^\xi = \mathrm{T}^\xi_* \colon \partial\xi_0 \mapsto \partial\xi_0 + \partial\xi \tag{7.24}$$

The morphism Π is translation-invariant

$$\Pi_j \mathrm{T}^\xi_* = \mathrm{T}^\xi \Pi_j \qquad \xi \in Q \tag{7.25}$$

7.6 Pull back the skew form $I\langle\, \xi_1, \xi_2 \,\rangle$ to a skew form $I_Q\langle\, \eta_1, \eta_2 \,\rangle$ on $\oplus_j \mathcal{D}^{\text{int}}_j(Q)$

Pull back the skew form $I\langle\, \xi_1, \xi_2 \,\rangle$ from $\oplus_j Q_j$ along the morphism Π to get a skew form $I_Q\langle\, \eta_1, \eta_2 \,\rangle$ on $\mathcal{D}^{\text{int}}(Q) = \oplus_j \mathcal{D}^{\text{int}}_j(Q)$ defined almost everywhere

$$\begin{aligned} &I_Q\langle\, \eta_1, \eta_2 \,\rangle = \Pi^* I\langle\, \eta_1, \eta_2 \,\rangle = I\langle\, \Pi_{j_1}\eta_1, \Pi_{j_2}\eta_2 \,\rangle \qquad \eta_1, \eta_2 \in \oplus_j \mathcal{D}^{\text{int}}_j(Q) \\ &I_Q \in \mathrm{Hom}(\oplus_j \mathcal{D}^{\text{int}}_j(Q) \times \oplus_j \mathcal{D}^{\text{int}}_j(Q), \mathbb{Z}) \quad \text{(defined almost everywhere)} \end{aligned} \tag{7.26}$$

satisfying (wherever defined)

$$I_Q\langle\, \eta_1, \eta_2 \,\rangle \neq 0 \quad \text{only if } \deg(\eta_1) + \deg(\eta_2) = 2 \tag{7.27}$$

$$I_Q\langle\, \eta_1, \eta_2 \,\rangle = -I_Q\langle\, \eta_2, \eta_1 \,\rangle \tag{7.28}$$

$$I_Q\langle\, \partial\eta_1, \eta_2 \,\rangle = -I_Q\langle\, \eta_1, \partial\eta_2 \,\rangle \tag{7.29}$$

$$I_Q\langle\, \eta_1, \mathrm{T}^\xi_*\eta_2 \,\rangle = I_Q\langle\, \eta_1, \eta_2 \,\rangle \qquad \deg(\eta_2) \geq 1 \qquad \xi \in Q \tag{7.30}$$

7.7 Pull back J along $d\Pi_1$

The morphism map Π_1 is $\Pi_{1,n-1} \colon \mathcal{D}^{\text{int}}_1(Q) \to \mathcal{D}^{\text{int}}_n(M)$. Its derivative identifies the tangent spaces $T_\xi Q$ with $\mathcal{D}_1 = \mathcal{D}_n(M)$ so J acts on the $T_\xi Q$ and on the dual cotangent spaces $T^*_\xi Q$

$$d\Pi_1 \colon T_\xi Q \to \mathcal{D}_1 \qquad J d\Pi_1 = d\Pi_1 J \qquad J d\Pi^*_1 = d\Pi^*_1 J \tag{7.31}$$

The J operator thereby acts on $\Omega_1(Q)$, the space of 1-forms on Q, and on the 1-currents $\mathcal{D}_1(Q)$

$$J\colon \Omega_1(Q) \to \Omega_1(Q) \qquad J\colon \mathcal{D}_1(Q) \to \mathcal{D}_1(Q) \tag{7.32}$$

inheriting the properties (7.13–7.15)

$$J^2 = -1 \quad \text{on } \mathcal{D}_1(Q) \tag{7.33}$$

$$I_Q\langle\, J\eta_1, J\eta_2 \,\rangle = I\langle\, \eta_1, \eta_2 \,\rangle \qquad \eta_1, \eta_2 \in \mathcal{D}_1(Q) \tag{7.34}$$

$$I_Q\langle\, \eta, J\eta \,\rangle \geq 0 \qquad \eta \neq 0 \tag{7.35}$$

where again the last property makes sense on an appropriate extension of $\mathcal{D}_1(Q)$.

7.8 Cauchy-Riemann equations on Q

Now we have the ingredients to write Cauchy-Riemann equations on Q analogous to the Cauchy-Riemann equations on a Riemann surface Σ in the form of equations (5.7-5.8). A fundamental solution $F_Q(\eta_0, \eta_1)$ is a complex bilinear form (almost everywhere defined)

$$F_Q \in \mathrm{Hom}(\mathcal{D}^{\mathrm{int}}_0(Q) \times \mathcal{D}^{\mathrm{int}}_1(Q), \mathbb{C}) \tag{7.36}$$

$$F_Q(\eta_0, \partial\eta_2) = 2\pi i I_Q\langle\, \eta_0, \eta_2 \,\rangle \qquad \eta_0 \in \mathcal{D}^{\mathrm{int}}_0(Q) \quad \eta_2 \in \mathcal{D}^{\mathrm{int}}_2(Q) \tag{7.37}$$

$$F_Q(\eta_0, (J-i)\eta_1) = 0 \qquad \eta_0 \in \mathcal{D}_0(Q, \mathbb{C}) \quad \eta_2 \in \mathcal{D}_2(Q, \mathbb{C}) \tag{7.38}$$

where in the last equation $F_Q(\eta_0, \eta_1)$ is extended by linearity to the complex currents $\mathcal{D}_j(Q, \mathbb{C}) = \mathcal{D}_j(Q) \otimes \mathbb{C}$.

There is the possibility of imposing the additional condition that the fundamental solution $F_Q(\eta_0, \eta_1)$ is the pullback of a form $F(\xi_0, \xi_1)$ on $Q_0 \times Q_1$ (which amounts to a translation-invariance condition)

$$F \in \mathrm{Hom}(Q_0 \times Q_1, \mathbb{C}), \qquad F_Q = \Pi^* F, \qquad F_Q(\eta_0, \eta_1) = F(\Pi_0\eta_0, \Pi_1\eta_1) \tag{7.39}$$

$$F(\xi_0, \partial\xi_2) = 2\pi i I\langle\, \xi_0, \xi_2 \,\rangle \tag{7.40}$$

$$F(\xi_0, (J-i)\xi_1) = 0 \tag{7.41}$$

8 The complex examples $Q(M)$ for n even or odd

Now let n be even or odd. The operator $J = \epsilon_n *$ is imaginary when n is even so the tangent spaces $T_\xi Q$ will have to be complex in order for J to act. Let Q be one of the metric abelian groups

$$Q = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}^{\mathrm{int}}_n(M) \qquad \partial\xi_0 \in \partial\mathcal{D}^{\mathrm{int}}_{n-1}(M) \tag{8.1}$$

When n is odd complex conjugation will be an automorphism, which will allow restricting to the real part of Q, recovering the real examples described in the previous section.

Now let the chain complex $\oplus_j Q_j$ be

$$0 \xrightarrow{\partial} Q_3 \xrightarrow{\partial} Q_2 \xrightarrow{\partial} Q_1 \xrightarrow{\partial} Q_0 \xrightarrow{\partial} Q_{-1} \xrightarrow{\partial} 0 \tag{8.2}$$

$$
\begin{aligned}
&Q_0 = Q \qquad Q_1 = \mathcal{D}^{\text{int}}_n(M) \oplus i\mathcal{D}^{\text{int}}_n(M) \qquad Q_2 = \left[\mathcal{D}^{\text{int}}_{n+1}(M) \oplus i\mathcal{D}^{\text{int}}_{n+1}(M)\right]/{Q_0}^{\perp} \\
&Q_{-1} = \mathbb{Z}\partial\xi_0 \qquad Q_3 = \left[\mathcal{D}^{\text{int}}_{n+2}(M) \oplus i\mathcal{D}^{\text{int}}_{n+2}(M)\right]/{Q_{-1}}^{\perp} \qquad Q_{-1} \cong \mathbb{Z} \qquad Q_3 \cong \mathbb{Z} \oplus i\mathbb{Z}
\end{aligned} \tag{8.3}
$$

where $Q_0^{\perp}$ and $Q_{-1}^{\perp}$ are the orthogonal complements now in the skew-hermitian intersection form $I_M\langle\,\bar{\xi}_1, \xi_2\,\rangle$ defined in equation (4.5)

$$
Q_0^{\perp} = \left\{\xi \in \mathcal{D}^{\text{int}}_{n+1}(M) \oplus i\mathcal{D}^{\text{int}}_{n+1}(M) :\ I_M\langle\,\bar{\xi}', \xi\,\rangle = 0 \quad \forall \xi' \in Q_0\right\} \tag{8.4}
$$

$$
Q_{-1}^{\perp} = \left\{\xi \in \mathcal{D}^{\text{int}}_{n+2}(M) \oplus i\mathcal{D}^{\text{int}}_{n+2}(M) :\ I_M\langle\,\bar{\xi}', \xi\,\rangle = 0 \quad \forall \xi' \in Q_{-1}\right\} \tag{8.5}
$$

The vector space $\mathcal{D}_1$ of infinitesimal elements of Q_1 is now the complex vector space

$$
\mathcal{D}_1 = \text{Hom}(Q_1, \mathbb{R})^* = \mathcal{D}^{\text{int}}_n(M) \oplus i\mathcal{D}^{\text{int}}_n(M) \tag{8.6}
$$

so $J = \epsilon_n *$ acts on $\mathcal{D}_1$ for n even or odd. By construction, $I_M\langle\,\bar{\xi}_1, \xi_2\,\rangle$ gives an almost everywhere defined nondegenerate skew-hermitian form $I\langle\,\bar{\xi}_1, \xi_2\,\rangle$ on $\oplus_j Q_j$ with values in $\mathbb{Z} \oplus i\mathbb{Z}$

$$
I \in \text{Hom}(\overline{\oplus_{j_1} Q_{j_1}} \times \oplus_{j_2} Q_{j_2}, \mathbb{Z} \oplus i\mathbb{Z}) \tag{8.7}
$$

satisfying properties inherited from properties (4.6-4.12) of $I_M\langle\,\bar{\xi}_1, \xi_2\,\rangle$.

Again there is a morphism of chain complexes of metric abelian groups

$$
\begin{tikzcd}
\mathcal{D}^{\text{int}}_4(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_4"] & \mathcal{D}^{\text{int}}_3(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_3"] & \mathcal{D}^{\text{int}}_2(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_2"] & \mathcal{D}^{\text{int}}_1(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_1"] & \mathcal{D}^{\text{int}}_0(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_0"] & \mathcal{D}^{\text{int}}_{-1}(Q) \arrow[r] \arrow[d, "\Pi_{-1}"] & 0 \\
0 \arrow[r] & Q_3 \arrow[r, "\partial"] & Q_2 \arrow[r, "\partial"] & Q_1 \arrow[r, "\partial"] & Q_0 \arrow[r, "\partial"] & Q_{-1} \arrow[r] & 0
\end{tikzcd} \tag{8.8}
$$

as in equations (7.16-7.22) for the real case. Again Π_1 induces isomorphisms of vector spaces

$$
d\Pi_1 \colon T_\xi Q \to \mathcal{D}_1\,, \qquad d\Pi_1^* \colon \Omega_1 \to T^*_\xi Q \tag{8.9}
$$

so J acts on $T_\xi Q$ and on $T^*_\xi Q$ and on $\Omega_1(Q)$. Again the skew-hermitian form $I\langle\,\bar{\xi}_1, \xi_2\,\rangle$ pulls back to a skew-hermitian form $I_Q\langle\,\bar{\eta}_1, \eta_2\,\rangle$ on $\oplus_j \mathcal{D}^{\text{int}}_j(Q)$. Again the properties (4.6-4.12) are inherited from M. Again Cauchy-Riemann equations on Q can be expressed in terms of J and $I_Q\langle\,\bar{\eta}_1, \eta_2\,\rangle$.

For every complex quasi Riemann surface there is an underlying real quasi Riemann surface which has the same morphism (8.8) of chain complexes, the same J operator, and the real part $\text{Re}\, I\langle\,\bar{\xi}_1, \xi_2\,\rangle$ as the skew form on $\oplus_j Q_j$. The complex quasi Riemann surfaces are the real quasi Riemann surfaces with additional structure: multiplication by i and complex conjugation.

9 Definition of quasi Riemann surface

Finally we try to define *quasi Riemann surface*. The definition should encompass all the examples $Q(M)$ as narrowly as possible, providing structure sufficient to express Cauchy-Riemann equations.

A *complex* quasi Riemann surface is a metric abelian group Q along with a morphism $Q \xrightarrow{\partial} \mathbb{Z}$ and an automorphism $\xi \mapsto \bar{\xi}$ (called complex conjugation) with $\partial\bar{\xi} = \partial\xi$. A *real* quasi Riemann surface is one where complex conjugation acts trivially $\bar{\xi} = \xi$.

Let the augmented integral chain complex of Q be

$$\cdots \xrightarrow{\partial} \mathcal{D}_3^{\text{int}}(Q) \xrightarrow{\partial} \mathcal{D}_2^{\text{int}}(Q) \xrightarrow{\partial} \mathcal{D}_1^{\text{int}}(Q) \xrightarrow{\partial} \mathcal{D}_0^{\text{int}}(Q) \xrightarrow{\partial} \mathcal{D}_{-1}^{\text{int}}(Q) \xrightarrow{\partial} 0 \tag{9.1}$$

$$\mathcal{D}^{\text{int}}(Q) = \bigoplus_{j=-1} \mathcal{D}_j^{\text{int}}(Q) \qquad \mathcal{D}_{-1}^{\text{int}}(Q) = \mathbb{Z} \qquad \partial\delta_\xi = \partial\xi \quad \xi \in Q \tag{9.2}$$

Complex conjugation acts on $\mathcal{D}^{\text{int}}(Q)$ as the push-forward of complex conjugation on Q.

There is a skew-hermitian form

$$I_Q \in \text{Hom}(\overline{\mathcal{D}^{\text{int}}(Q)} \times \mathcal{D}^{\text{int}}(Q), \mathbb{A}) \qquad \mathbb{A} = \mathbb{Z} \oplus i\mathbb{Z} \text{ (complex case) or } \mathbb{Z} \text{ (real case)} \tag{9.3}$$

defined almost everywhere, satisfying

$$I_Q\langle\, \bar\eta_1, \eta_2 \,\rangle \neq 0 \quad \text{only if } \deg(\eta_1) + \deg(\eta_2) = 2 \tag{9.4}$$

$$I_Q\langle\, \bar\eta_1, \eta_2 \,\rangle = -\overline{I_Q\langle\, \bar\eta_2, \eta_1 \,\rangle} \tag{9.5}$$

$$I_Q\langle\, \overline{\partial\eta_1}, \eta_2 \,\rangle = -I_Q\langle\, \bar\eta_1, \partial\eta_2 \,\rangle \tag{9.6}$$

$$I_Q\langle\, \bar\eta_1, \mathrm{T}_*^\xi\eta_2 \,\rangle = I_Q\langle\, \bar\eta_1, \eta_2 \,\rangle \qquad \deg(\eta_2) \geq 1 \qquad \xi \in Q\,. \tag{9.7}$$

There is a linear operator J equivalently described as

$$J \colon T_\xi Q \to T_\xi Q \qquad J \colon T_\xi^* Q \to T_\xi^* Q \qquad \xi \in Q \tag{9.8}$$

$$J \colon \mathcal{D}_1(Q) \to \mathcal{D}_1(Q) \qquad J \colon \Omega_1(Q) \to \Omega_1(Q) \tag{9.9}$$

satisfying on the tangent spaces $T_\xi(Q)$

$$J^2 = -1 \tag{9.10}$$

$$I_Q\langle\, \overline{J\eta_1}, J\eta_2 \,\rangle = I_Q\langle\, \bar\eta_1, \eta_2 \,\rangle \tag{9.11}$$

$$I_Q\langle\, \bar\eta, J\eta \,\rangle > 0 \qquad \eta \neq 0 \tag{9.12}$$

$$JT_*^\xi = T_*^\xi J \tag{9.13}$$

Given this structure we define metric abelian groups

$$Q_j = \mathcal{D}_j^{\text{int}}(Q)/\mathcal{D}_{2-j}^{\text{int}}(Q)^\perp \qquad j = -1, 0, 1, 2, 3 \tag{9.14}$$

so there is a morphism of chain complexes

$$\begin{tikzcd}
\cdots \mathcal{D}_4^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_4"] & \mathcal{D}_3^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_3"] & \mathcal{D}_2^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_2"] & \mathcal{D}_1^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_1"] & \mathcal{D}_0^{\text{int}}(Q) \arrow[r, "\partial"] \arrow[d, "\Pi_0"] & \mathcal{D}_{-1}^{\text{int}}(Q) \arrow[r] \arrow[d, "\Pi_{-1}"] & 0 \\
0 \arrow[r] & Q_3 \arrow[r, "\partial"] & Q_2 \arrow[r, "\partial"] & Q_1 \arrow[r, "\partial"] & Q_0 \arrow[r, "\partial"] & Q_{-1} \arrow[r] & 0
\end{tikzcd} \tag{9.15}$$

with $\Pi_j = 0$, $j \geq 4$. Impose as additional condition on $I_Q\langle\, \bar\eta_1, \eta_2 \,\rangle$

$$Q_0 = Q \qquad \Pi_0\delta_\xi = \xi \qquad \xi \in Q \tag{9.16}$$

$I_Q\langle\,\bar\eta_1,\eta_2\,\rangle$ descends to a nondegenerate skew-hermitian form $I\langle\,\bar\xi_1,\xi_2\,\rangle$ on $\oplus_j Q_j$ and J acts on the vector space $\mathcal{D}_1$ of infinitesimal elements of Q_1.

Finally, the morphisms

$$\Pi_{j,k}\colon \mathcal{D}^{\mathrm{int}}_j(\mathcal{D}^{\mathrm{int}}_k Q)) \to\colon \mathcal{D}^{\mathrm{int}}_{j+k}(Q) \tag{9.17}$$

should descend to morphisms

$$\Pi^Q_{j,k}\colon \mathcal{D}^{\mathrm{int}}_j(Q_k) \to Q_{j+k} \qquad \Pi^Q_{j,0} = \Pi_j \tag{9.18}$$

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