Quasi Riemann surfaces

II. Questions, comments, speculations

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Abstract

This is the second note of a series on *quasi Riemann surfaces* which are metric abelian groups whose integral currents are analogous to the integral currents in a Riemann surface. This note is a collection of questions, comments, and speculations.

Contents

Quasi Riemann surfaces were described in an earlier paper [1] motivated by considerations from quantum field theory. The main elements of the mathematical proposal are summarized in [2]. The present note follows with a collection of questions, comments, and speculations. These are in addition to the many questions implicit in [2] about making its naive formal constructions mathematically solid. The notation is as in [2]. References to equations in [2] are of the form (I-6.2). As in [2] the presentation is entirely naive and formal; there is no attempt at rigor; there is no attempt to be precise about topologies or domains of definition.

1 Classification conjecture

> **Conjecture:** Connected quasi Riemann surfaces Q are classified by their Jacobians $J(Q)$. Every quasi Riemann surface Q is isomorphic to the quasi Riemann surface $Q(\Sigma)$ associated to a unique two-dimensional conformal space Σ whose ordinary Jacobian $J(\Sigma)$ is isomorphic to $J(Q)$.

The following sections elaborate on this conjecture. But the meaning of "isomorphic" is left vague. There is no supporting evidence so it would better be called speculation or wishful thinking. Its appeal is the nice structure that it implies (described in sections 8 and 9 below).

The conjecture would open a route to a new construction of quantum field theories on conformal $2n$-manifolds M. For each two-dimensional quantum field theory on Riemann surfaces Σ we would first have to generalize its quantum fields from the points in Σ to the integral 0-currents in Σ. This is a problem in two-dimensional quantum field theory. Then we could use the conjectured isomorphisms to transport the correlation functions from $Q(\Sigma) = \mathcal{D}_0^{\text{int}}(\Sigma)$ to the $Q(M)$.

2 Homology groups

The chain complex of metric abelian groups (I-7.2), (I-8.2), (I-9.15)

$$0 \xrightarrow{\partial} Q_3 \xrightarrow{\partial} Q_2 \xrightarrow{\partial} Q_1 \xrightarrow{\partial} Q_0 \xrightarrow{\partial} Q_{-1} \xrightarrow{\partial} 0 \qquad Q_0 = Q \qquad Q_{-1} = \mathbb{Z} \tag{2.1}$$

has homology groups

$$H_j = Q_{j,0}/\partial Q_{j+1} \qquad Q_{j,0} = \operatorname{Ker} \partial \subset Q_j \qquad H_3 = H_{-1} = 0 \tag{2.2}$$

The skew(-hermitian) form $I\langle \bar{\xi}_1, \xi_2 \rangle$ on $\oplus_j Q_j$ descends to give a nondegenerate form on the $\oplus_j H_j$ which in the real case is skew with values in $\mathbb{Z}$ and in the complex case is skew-hermitian with values in $\mathbb{Z} \oplus i\mathbb{Z}$. The skew(-hermitian) form provides Poincaré duality for the homology groups H_j.

In the real examples $Q(M)$ with $Q = \mathcal{D}_{n-1}^{\text{int}}(M)_{\mathbb{Z}\partial\xi_0}$ for M a conformal manifold of dimension $2n$ with n odd and $n > 1$

$$H_0 = H_{n-1}(M) \qquad H_1 = H_n(M) \qquad H_2 = H_{n+1}(M) \tag{2.3}$$

In the complex examples $Q = \mathcal{D}_{n-1}^{\text{int}}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}_n^{\text{int}}(M)$ with n even or odd

$$H_0 = H_{n-1}(M) \qquad H_1 = H_n(M) \oplus iH_n(M) \qquad H_2 = H_{n+1}(M) \tag{2.4}$$

In the real examples $Q(\Sigma) = \mathcal{D}_0^{\mathrm{int}}(\Sigma)$ for Σ a Riemann surface, i.e., $n = 1$,

$$H_0 = H_0^{red}(\Sigma) = H_0(\Sigma)/\mathbb{Z} \qquad H_1 = H_1(\Sigma) \qquad H_2 = H_2^{red}(\Sigma) = H_2(\Sigma)/\mathbb{Z} \tag{2.5}$$

because the chain complex (2.1) is the *augmented* chain complex of integral currents in Σ whose homology is the reduced homology.

3 Connectedness

For simplicity, we assume the connectedness condition

$$H_0 = 0 \tag{3.1}$$

Equivalently the metric abelian subgroup $Q_{0,0} = \partial^{-1}\{0\} \subset Q$ is connected as a topological space. In the examples $Q(M)$ with $n > 1$ the connectedness condition is

$$H_{n-1}(M) = 0 \tag{3.2}$$

In the examples $Q(\Sigma)$ for Σ a Riemann surface the condition is

$$H_0^{red}(\Sigma) = 0 \tag{3.3}$$

which is the condition that Σ is connected.

Given the connectedness condition $H_0 = 0$, Poincaré duality implies

C1 The only nonzero homology is in the middle dimension

$$H_j = 0 \quad j \neq 1 \qquad \oplus_j H_j = H_1 \tag{3.4}$$

C2 H_1 is torsion-free.

4 The Jacobian $J(Q)$

The real vector space of j-forms is $\Omega_j = \mathrm{Hom}(Q_j, \mathbb{R})$. Its dual $\mathcal{D}_j = \Omega_j^*$ is the vector space of j-currents. The $\mathcal{D}_j$ form a chain complex

$$0 \xrightarrow{\partial} \mathcal{D}_3 \xrightarrow{\partial} \mathcal{D}_2 \xrightarrow{\partial} \mathcal{D}_1 \xrightarrow{\partial} \mathcal{D}_0 \xrightarrow{\partial} \mathcal{D}_{-1} \xrightarrow{\partial} 0 \tag{4.1}$$

The linear operator J acts on $\mathcal{D}_1$ satisfying $J^2 = -1$. In the examples $J = \epsilon_n *$ with $\epsilon_n^2 = (-1)^{n-1}$. $\mathcal{D}_1$ is a Hilbert space with inner product $\langle \bar{\xi}_1, \xi_2 \rangle = I\langle \bar{\xi}_1, J\xi_2 \rangle$. $\mathcal{D}_1$ decomposes into orthogonal subspaces

$$\begin{gathered} \mathcal{D}_1 = \partial\mathcal{D}_2 \oplus \mathcal{D}_1^{\mathrm{har}} \oplus J\partial\mathcal{D}_2 \qquad \mathcal{D}_1^{\mathrm{har}} = \mathrm{Ker}\,\partial \cap J\,\mathrm{Ker}\,\partial \\ \mathrm{Ker}\,\partial = \partial\mathcal{D}_2 \oplus \mathcal{D}_1^{\mathrm{har}} \qquad \mathrm{Ker}(\partial J) = J\,\mathrm{Ker}\,\partial = \mathcal{D}_1^{\mathrm{har}} \oplus J\partial\mathcal{D}_2 \end{gathered} \tag{4.2}$$

$\mathcal{D}_1^{\mathrm{har}}$ is the subspace of harmonic 1-currents. It represents the real homology and contains the integer homology H_1 as a lattice

$$H_1 \subset \mathcal{D}_1^{\mathrm{har}} \qquad \mathcal{D}_1^{\mathrm{har}} = H_1^{\mathbb{R}} = \mathbb{R} \otimes H_1 \tag{4.3}$$

$\mathcal{D}_1^{\text{har}}$ is a complex Hilbert space with complex structure J.

The *Jacobian* $J(Q)$ is the lattice H_1 in the complex Hilbert space $\mathcal{D}_1^{\text{har}}$ along with the skew(-hermitian) intersection form $I\langle\,\bar{\xi}_1, \xi_2\,\rangle$ on H_1 which extends to $\mathcal{D}_1^{\text{har}}$ to give the Hilbert space inner product $\langle\,\bar{\xi}_1, \xi_2\,\rangle = I\langle\,\bar{\xi}_1, J\xi_2\,\rangle$.

Call the complex dimension of $\mathcal{D}_1^{\text{har}}$ the *genus* g of the quasi Riemann surface

$$\operatorname{rank}_{\mathbb{Z}}(H_1) = 2g \qquad \dim_{\mathbb{R}}(\mathcal{D}_1^{\text{har}}) = 2g \qquad \dim_{\mathbb{C}}(\mathcal{D}_1^{\text{har}}) = g \tag{4.4}$$

In the real examples $Q(M)$ for M a conformal $2n$-manifold with n odd

$$H_1 = H_n(M) \qquad \mathcal{D}_1^{\text{har}} = H_n(M,\mathbb{R}) \qquad g = \frac{1}{2}b_n \tag{4.5}$$

where b_n the nth Betti number. In the complex examples where n is odd or even

$$H_1 = H_n(M) \oplus iH_n(M) \qquad \mathcal{D}_1^{\text{har}} = H_n(M,\mathbb{R}) \oplus H_n(M,\mathbb{R}) = H_n(M,\mathbb{C}) \qquad g = b_n \tag{4.6}$$

We write $J(Q(M))$ for the Jacobian $J(Q_{\mathbb{Z}\partial\xi_0})$ of any of the quasi Riemann surfaces $Q(M)$.

In the example $Q(\Sigma)$ for Σ an ordinary Riemann surface the Jacobian of the quasi Riemann surface is identical to the ordinary Jacobian $J(\Sigma)$ of the Riemann surface

$$J(Q(\Sigma)) = J(\Sigma) \tag{4.7}$$

5 What weak sense of isomorphism?

For the classification conjecture to be possible, isomorphism of quasi Riemann surfaces Q and Q' must be weaker than metric or topological equivalence. The Almgren-Dold-Thom isomorphism [3]

$$\pi_j(\mathcal{D}_k^{\text{int}}(M)_0 = H_{j+k}(M) \qquad j \geq 1 \tag{5.1}$$

implies

$$\pi_j(\mathcal{D}_{n-1}^{\text{int}}(M)_{\mathbb{Z}\partial\xi_0}) = H_{j+n-1}(M) \qquad j \geq 1 \tag{5.2}$$

so the metric topology of $Q(M)$ depends on more of the structure of M than the Jacobian $J(Q(M))$. (The Almgren-Dold-Thom isomorphism is presumably the composition

$$\pi_j(\mathcal{D}_k^{\text{int}}(M)_0) \xrightarrow{h_*} H_j(\mathcal{D}_k^{\text{int}}(M)_0) \xrightarrow{\Pi_*} H_{j+k}(M) \qquad j \geq 1 \tag{5.3}$$

of the Hurewicz homomorphism h_* with the action of $\Pi_{j,k}$ on the homology.)

We might define a *weakening* of a quasi Riemann surface Q to be a quasi Riemann surface Q' along with a homomorphism

$$f\colon Q \to Q' \tag{5.4}$$

satisfying

W1 f is continuous in the metric topology or perhaps is a metric map.

W2 $f(Q)$ is dense in Q'.

W3 f is injective or at least its kernel is insignificant.

and preserving the quasi Riemann surface structure

W4 $\quad I_Q = f_* I_{Q'}$

W5 $\quad df\, J = J df$

Then we might suppose that there is a weakest Q^{weak} such that Q^{weak} is a weakening of every weakening of Q. Finally we might suppose that Q^{weak} depends on the Jacobian $J(Q)$. We might call the metric on Q induced from Q^{weak} the *weak metric*, so Q^{weak} is the completion of Q in the weak metric. The weak metric sees only the structure given by I_Q and J of the integral j-currents in Q for $j = 0, 1, 2, 3$. The classification conjecture could hold in the sense that any two quasi Riemann surfaces with isomorphic Jacobians are isomorphic as metric abelian groups in their weak metrics.

6 What two-dimensional conformal spaces?

The conjecture would go on to say that Q^{weak} is isomorphic to $Q(\Sigma)$ for a unique two-dimensional conformal space Σ with the same Jacobian as Q.

There are at least two problems with this part of the conjecture:

1. First, the two-dimensional conformal spaces must be more general than the Riemann surfaces since not every Jacobian is the Jacobian of a Riemann surface.

2. The morphism $\Pi_3 \colon \mathcal{D}_3^{\text{int}}(Q) \to Q_3 = \mathbb{Z}$ should be surjective. It is surjective in all the examples $Q(M)$ with $n > 1$.

For Σ a Riemann surface and $Q = Q(\Sigma) = \mathcal{D}_0^{\text{int}}(\Sigma)$, the map $\Pi_3 \colon \mathcal{D}_3^{\text{int}}(Q) \to \mathcal{D}_3^{\text{int}}(\Sigma) = \mathbb{Z}$ is identically zero because the augmentation $\mathcal{D}_3^{\text{int}}(\Sigma) = \mathbb{Z}$ is artificial; there are actually no 3-currents in a Riemann surface.

We might define a two-dimensional conformal space to be a metric space Σ such that $Q_j = \mathcal{D}_j^{\text{int}}(\Sigma)$ is a quasi Riemann surface, with skew intersection form I_Σ and J operator acting in the middle dimension. Such Σ would have to satisfy

$$\mathcal{D}_j^{\text{int}}(\Sigma) = 0 \quad j > 3 \qquad \mathcal{D}_3^{\text{int}}(\Sigma) = \mathbb{Z} \tag{6.1}$$

Then we might ask

1. Are there such spaces Σ?

2. Can an ordinary Riemann surface be augmented in some sense to give such a Σ?

3. Is there a unique such Σ for every Jacobian?

4. Is every $Q(\Sigma)$ maximally weak? i.e., does $Q(\Sigma)^{\text{weak}} = Q(\Sigma)$?

5. Does every Q^{weak} equal some $Q(\Sigma)$?

6. Is there a way to reconstruct a two-dimensional conformal space Σ from its quasi Riemann surface $Q(\Sigma)$ so that the same process might be applied to any Q to get a Σ with $Q^{\text{weak}} = Q(\Sigma)$?

7. Do such spaces Σ have enough in common with ordinary Riemann surfaces to support some form of complex analysis in one complex variable?

7 The automorphism group $\mathbf{Aut}(Q)$

Assume the classification conjecture. Assume every quasi Riemann surface is given its weak metric. Define the automorphism group $\mathbf{Aut}(Q)$ of a quasi Riemann surface to be the group of automorphisms of Q^{weak} as a metric abelian group that preserve the skew(-hermitian) form I_Q and the J operator. Given the conjecture, $\mathbf{Aut}(Q)$ will only depend on the Jacobian $J(Q)$ and will be isomorphic to $\mathbf{Aut}(Q(\Sigma))$ with $J(\Sigma) = J(Q)$.

Suppose M is a conformal manifold with the same Jacobian as Q, $J(Q_{\mathbb{Z}\partial\xi_0}) = J(Q)$. Let $\mathbf{Conf}(M)$ be the conformal symmetry group of M. Let $\mathbf{Conf}(M)_{\partial\xi_0}$ be the subgroup that leaves fixed the integral $(n-2)$-boundary $\partial\xi_0$. Then $\mathbf{Conf}(M)_{\partial\xi_0}$ will act as automorphisms of the quasi Riemann surface $Q_{\mathbb{Z}\partial\xi_0}$. By the conjecture $Q_{\mathbb{Z}\partial\xi_0}$ is isomorphic to Q so $\mathbf{Conf}(M)_{\partial\xi_0}$ occurs as a conjugation class of subgroups of $\mathbf{Aut}(Q)$

$$\mathbf{Conf}(M)_{\partial\xi_0} \subset \mathbf{Aut}(Q) \tag{7.1}$$

for all M with the given Jacobian and all $\partial\xi_0$. For example consider the trivial Jacobian $J(Q) = 0$ and $M = S^{2n}$. Let $\partial\xi_0$ be an $(n-2)$-sphere. Then $\mathbf{Conf}(M)_{\partial\xi_0} = O(n-1) \times O(n+2)$ All of these will occur as subgroups in $\mathbf{Aut}(Q)$.

Assume that each ordinary two-dimensional conformal field theory can be extended from ordinary Riemann surfaces to the quasi Riemann surfaces $Q(\Sigma)$. Then the automorphism group $\mathbf{Aut}(Q(\Sigma))$ will act by on the conformal field theory by symmetries. There will be a group homomorphism

$$\mathbf{Aut}(Q) \to \mathbf{Sym}(\mathbf{CFT_2}) \tag{7.2}$$

for every symmetry group $\mathbf{Sym}(\mathbf{CFT_2})$ of every two-dimensional conformal field theory $\mathbf{CFT_2}$.

8 The bundle $Q(M) \to B(M)$ of quasi Riemann surfaces

$Q(M) \to B(M)$ is the bundle of quasi Riemann surfaces $Q_{\mathbb{Z}\partial\xi_0} \to \mathbb{Z}\partial\xi_0$ associated to a conformal $2n$-manifold M. All of the $Q_{\mathbb{Z}\partial\xi_0} \subset Q(M)$ have the same Jacobian so all are isomorphic to $Q = Q(\Sigma)$ with the same Jacobian. The spaces of isomorphisms

$$F_{\mathbb{Z}\partial\xi_0} = \mathbf{Iso}(Q, Q_{\mathbb{Z}\partial\xi_0}) \tag{8.1}$$

form a principal fiber bundle with structure group $\mathbf{Aut}(Q)$

$$F(M) \to B(M) \tag{8.2}$$

$Q(M) \to B(M)$ is the associated bundle with respect to the action of $\mathbf{Aut}(Q)$ on Q.

9 A universal homogeneous bundle of quasi Riemann surfaces

For each Jacobian $J(Q)$, $Q = Q(\Sigma)$, we construct a universal homogeneous principal bundle

$$F^{\text{univ}}(Q) \to B^{\text{univ}}(Q) \tag{9.1}$$

such that, for every manifold M with the given Jacobian, $F(M)$ embeds naturally in $F^{\mathrm{univ}}(Q)$

$$\begin{array}{ccc} F(M) & \hookrightarrow & F^{\mathrm{univ}}(Q) \\ \downarrow & & \downarrow \\ B(M) & \hookrightarrow & B^{\mathrm{univ}}(Q) \end{array} \tag{9.2}$$

The construction is motivated by the fact that all of the $Q_{\mathbb{Z}\partial\xi_0} \subset Q(M)$ are actually *identical* in the middle degree

$$Q_{\mathbb{Z}\partial\xi_0,1} = \mathcal{D}_n^{\mathrm{int}}(M) \qquad \text{or} \qquad Q_{\mathbb{Z}\partial\xi_0,1} = \mathcal{D}_n^{\mathrm{int}}(M) \oplus i\mathcal{D}_n^{\mathrm{int}}(M) \tag{9.3}$$

Let $\mathbf{Aut}(Q_1)$ be the group of automorphisms of the metric abelian group Q_1 that preserve all the structure inherited from Q: the subgroups $\partial Q_2 \subset \operatorname{Ker}\partial \subset Q_1$, the skew(-hermitian) form $I\langle\, \bar{\xi}_1, \xi_2\,\rangle$ and the J operator. $\mathbf{Aut}(Q)$ is the subgroup

$$\mathbf{Aut}(Q) \subset \mathbf{Aut}(Q_1) \tag{9.4}$$

consisting of those $f \in \mathbf{Aut}(Q_1)$ for which $f_*\colon \mathcal{D}_1^{\mathrm{int}}(Q_1) \to \mathcal{D}_1^{\mathrm{int}}(Q_1)$ is compatible with $\Pi^Q_{1,1}\colon \mathcal{D}_1^{\mathrm{int}}(Q_1) \to Q_2$. The universal homogeneous bundle is

$$F^{\mathrm{univ}}(Q) \to B^{\mathrm{univ}}(Q) \qquad = \qquad \mathbf{Aut}(Q_1) \to \mathbf{Aut}(Q_1)/\,\mathbf{Aut}(Q) \tag{9.5}$$

To construct the embedding $F(M) \hookrightarrow F^{\mathrm{univ}}(Q)$ use the fact that all the $Q_{\mathbb{Z}\partial\xi_0,1}$ are the same to define

$$g\colon F(M) \times F(M) \to \mathbf{Aut}(Q_1) \qquad g(f_1, f_2) = f_1^{-1} \circ f_2/Q_1 \tag{9.6}$$

satisfying

$$g(f, f) = 1 \qquad g(f_1, f_2)g(f_2, f_3) = g(f_1, f_3) \tag{9.7}$$

Then for any $f_0 \in F(M)$

$$f \mapsto g(f_0, f) \tag{9.8}$$

embeds the principal bundle $F(M) \to B(M)$ into the universal homogeneous bundle.

Somewhat more detailed descriptions of $\mathbf{Aut}(Q_1) \to \mathbf{Aut}(Q_1)/\,\mathbf{Aut}(Q)$ and of the embedding $F(M) \hookrightarrow \mathbf{Aut}(Q_1)$ are in section 19 of [1].

10 Quasi holomorphic curves

Define a *quasi holomorphic curve* in a quasi Riemann surface Q to be a function $C\colon \Sigma \to Q$ from a Riemann surface Σ to Q that preserves the J operators and the skew-hermitian forms on integral currents

$$dCJ = JdC \qquad C^*I_Q = I_\Sigma \qquad I_\Sigma\langle\, \bar{\eta}_1, \eta_2\,\rangle = I_Q\langle\, \overline{C_*\eta_1}, C_*\eta_2\,\rangle \tag{10.1}$$

so a solution of the Cauchy-Riemann equations on Q pulls back along C to a solution of the Cauchy-Riemann equations on Σ. Defne a *local quasi holomorphic curve* to be a quasi holomorphic curve where the Riemann surface Σ is an open disk D in the complex plane.

In the examples $Q(\Sigma) = \mathcal{D}_0^{\rm int}(\Sigma)$ for Σ a Riemann surface there is a canonical quasi holomorphic curve $C\colon z \in \Sigma \mapsto \delta_z \in \mathcal{D}_0^{\rm int}(\Sigma)$. Restricting C to any coordinate neighborhood in Σ becomes a local quasi holomorphic curve in $Q(\Sigma)$. More quasi holomorphic curves in $Q(\Sigma)$ would be obtained by composing with automorphisms of $Q(\Sigma)$.

The local quasi holomorphic curves are the local probes of the quasi Riemann surface. Solutions of the Cauchy-Riemann equations on the quasi Riemann surface are displayed as ordinary meromorphic functions on the local quasi holomorphic curves. In the proposed quantum field theories, the local quasi holomorphic curves would express the local interactions of the $(n-1)$-dimensional objects in terms of the operator product expansions of the ordinary two-dimensional conformal field theory on the local quasi holomorphic curves.

If there are enough local quasi holomorphic curves in Q, then functions, forms, conformal tensors, and the like on Q can be represented as coherent collections indexed by the quasi holomorphic curves in Q of ordinary functions, forms, and conformal tensors on ordinary Riemann surfaces. Local function theory on Q will be expressed in terms of function theory on the collection of local quasi holomorphic curves.

Some questions:

1. Do quasi holomorphic curves exist in a general quasi Riemann surface Q? Do they exist in the examples $Q(M)$? Do *local* quasi holomorphic curves exist?
2. Is the conjectured weak metric needed on a quasi Riemann surface for there to be quasi holomorphic curves?
3. Given the classification conjecture, if Q is isomorphic to $Q(\Sigma)$ there should be a quasi holomorphic curve $\Sigma \to Q$, unique up to automorphisms.
4. For a given quasi Riemann surface, can the space of quasi holomorphic curves be described? the space of local quasi holomorphic curves?
5. Are there enough local quasi holomorphic curves to distinguish solutions of the Cauchy Riemann equations on Q?
6. Are there necessary and sufficient coherence conditions for collections of meromorphic functions or conformal tensors on the quasi holomorphic curves?
7. Can quasi holomorphic curves be constructed explicitly in the basic cases $M = S^{2n} = \mathbb{R}^{2n} \cup \{\infty\}$ for Σ the unit disk or the Riemann sphere.

11 The level $N(Q)\mathbb{Z}$ of a quasi Riemann surface

Define the *level* of a quasi Riemann surface Q to be the abelian subgroup $N(Q)\mathbb{Z} \subset \mathbb{Z}$ (written $N\mathbb{Z}$ when Q is understood) such that the skew(-hermitian) intersection form $I\langle \bar{\xi}_1, \xi_2 \rangle$ takes $\bar{Q}_{-1} \times Q_3$ to $N\mathbb{Z} \subset \mathbb{Z}$ (or to $N\mathbb{Z} \oplus iN\mathbb{Z} \subset \mathbb{Z} \oplus i\mathbb{Z}$ in the complex case).

$$I(\bar{Q}_{-1} \times Q_3) = \begin{cases} N\mathbb{Z} \subset \mathbb{Z} & \text{real case} \\ N\mathbb{Z} \oplus iN\mathbb{Z} \subset \mathbb{Z} \oplus i\mathbb{Z} & \text{complex case} \end{cases} \tag{11.1}$$

$N(Q)\mathbb{Z}$ is an invariant of the quasi Riemann surface. Up to now we have been implicitly assuming $N(Q) = 1$.

In the examples $Q(\Sigma) = \mathcal{D}_0^{\mathrm{int}}(\Sigma)$ for Σ a Riemann surface, the level is $N = 1$ because $\partial\delta_z = 1$, $\partial 1 = \Sigma$, and δ_z has intersection number 1 with Σ.

In the examples $Q(M)$ the level of $Q_{\mathbb{Z}\partial\xi_0}$ is the subgroup $N\mathbb{Z} \subset \mathbb{Z}$ of all the intersection numbers of $\partial\xi_0$ with integral $(n+2)$-currents in M. Call this invariant $N(\partial\xi_0)\mathbb{Z}$. It seems plausible that $\partial\xi_0/N$ will be an integral current. Then $\partial\xi_0/N$ will be a boundary if $H_{n-2}(M)$ is torsion-free.

For any quasi Riemann surface Q and any $N'\mathbb{Z}$ there is a quasi Riemann surface

$$Q_{N'\mathbb{Z}} = \partial^{-1}(N'\mathbb{Z}) \subset Q \qquad N(Q_{N'\mathbb{Z}}) = N'N(Q) \tag{11.2}$$

along with a net of inclusions

$$Q_{N_1\mathbb{Z}} \hookrightarrow Q_{N_2\mathbb{Z}} \qquad N_1\mathbb{Z} \subset N_2\mathbb{Z} \tag{11.3}$$

A more refined form of the classification conjecture would be that any Q of level $N\mathbb{Z}$ is isomorphic to $Q(\Sigma)_{N\mathbb{Z}}$ for the appropriate Σ.

12 The base space $B(M)$

The base space $B(M)$ of the bundle $Q(M) \to \mathcal{B}(M)$ of quasi Riemann surfaces

$$B(M) = \left\{ \textit{maximal } \mathbb{Z}\partial\xi_0 \subset \mathcal{D}_{n-2}^{\mathrm{int}}(M) \right\} \tag{12.1}$$

which perhaps can also be described as

$$B(M) = \text{ the image of } \partial\mathcal{D}_{n-1}^{\mathrm{int}}(M) - \{0\} \text{ in the projective space } P\partial\mathcal{D}_{n-1}^{\mathrm{distr}}(M)(\mathbb{R}) \tag{12.2}$$

When the homology group $H_{n-2}(M)$ is torsion-free, Every $\partial\xi_0'$ is of the form $\partial\xi_0' = N\partial\xi_0$ for $N(\partial\xi_0) = 1$, so all the $Q_{\mathbb{Z}\partial\xi_0}$ are of level 1 for $\mathbb{Z}\partial\xi_0 \in \mathcal{B}(M)$. We can restrict the definition of quasi Riemann surfaces requiring level 1. The bundle $F(M) \to B(M)$ is a principal fiber bundle as described in section 8 above.

When $H_{n-2}(M)$ is not torsion-free, the picture is more complicated. The base $\mathcal{B}(M)$ has a stratification indexed by the $N\mathbb{Z} \subset \mathbb{Z}$

$$\begin{aligned} \mathcal{B}(M)_{N\mathbb{Z}} &= \{\mathbb{Z}\partial\xi_0 \in \mathcal{B}(M) \colon N(\partial\xi_0)\mathbb{Z} \supset N\mathbb{Z}\} \\ \mathcal{B}(M)_{N_1\mathbb{Z}} &\subset \mathcal{B}(M)_{N_2\mathbb{Z}} \qquad N_1\mathbb{Z} \supset N_2\mathbb{Z} \end{aligned} \tag{12.3}$$

The strata of the total space $F(M)$ might be written (in the real case) as the space of morphisms

$$F(M)_{N\mathbb{Z}} = \mathbf{Mor}(Q_{N\mathbb{Z}} \to \mathbb{Z},\, \mathcal{D}_{n-1}^{\mathrm{int}}(M) \to \partial\mathcal{D}_{n-1}^{\mathrm{int}}(M)) \tag{12.4}$$

with structure group $\mathbf{Aut}(Q_{N\mathbb{Z}})$.

13 Conformal-Hodge metric spaces

The quasi Riemann surfaces $Q(M)$ are constructed using only the integral currents in M, the conformal Hodge $*$-operator acting in the middle dimension, and the intersection form on integral currents. So the subject is not $2n$-dimensional conformal manifolds, but metric spaces with a $*$-operator acting on n-forms and a nondegenerate intersection form $I(\xi_1, \xi_2)$ on pairs of integral currents whose degrees add to $2n$, possessing the same properties (I-4.2) – (I-4.12) as in conformal $2n$-manifolds.

14 $\mathbb{Z} \oplus i\mathbb{Z}$ in the definition of complex quasi Riemann surface

It looks awkward in the definition of complex quasi Riemann surface in section 9 of [2] that the skew-hermitian form takes values in the abelian group $\mathbb{Z} \oplus i\mathbb{Z}$. Perhaps the best that can be said is that $\mathbb{Z} \oplus i\mathbb{Z}$ is the smallest abelian group that contains the infinite cyclic group $\mathbb{Z}$ and has automorphisms $m \mapsto im$, $i^2 = -1$, and $m \mapsto \bar{m}$, $\overline{im} = -im$. The Jacobians of the complex quasi Riemann surfaces are the Jacobians that are invariant under these two automorphisms. In the complex examples $Q(M)$, the abelians group Q_1, Q_2, and Q_3 have these automorphisms, but Q_0 and Q_{-1} only have complex conjugation, not multiplication by i. The definitions $Q_{-1} = \mathbb{Z}\partial\xi_0$ and $Q = Q_0 = \mathcal{D}^{\mathrm{int}}_{n-1}(M)_{\mathbb{Z}\partial\xi_0} \oplus i\partial\mathcal{D}^{\mathrm{int}}_n(M)$ are chosen as the minimal extensions of the real examples such that the tangent spaces $T_\xi Q$ become complex, allowing $J = \epsilon_n *$ to act.

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