682A. Take-home exam. Due May 12th.

Answer the following three questions (there are three pages). Please do not just give mathematical answers. Significant credit will be given for work accompanied by a physical explanation.

1. Generalize the scaling equations to the anisotropic Kondo model with an anisotropic interaction

$$H_I = J^x\sigma^x(0)S^x + J^y\sigma^y(0)S^y + J^z\sigma^z(0)S^z \quad (1)$$

where $\sigma^a(0) = \sum_{k\alpha,k'\beta} c^\dagger{}_{k\alpha}\sigma^a_{\alpha\beta}c_{k'\beta}$ is the local spin density of the conduction sea and $S^{x,y,z}$ are the three components of the localized magnetic moment.

(a) Show that the scaling equations take the form

$$\frac{\partial J_a}{\partial \ln D} = -2J_bJ_c\rho + O(J^3),$$

where and (a, b, c) are a cyclic permutation of (x, y, z).

(b) Show that in the special case where $J_x = J_y = J_\perp$, the scaling equations become

$$\begin{aligned} \frac{\partial J_\perp}{\partial \ln D} &= -2J_zJ_\perp\rho + O(J^3), \\ \frac{\partial J_z}{\partial \ln D} &= -2(J_z)^2\rho + O(J^3), \end{aligned} \quad (2)$$

so that $J_z^2 - J_\perp^2 =$ constant.

(c) *Sketch* the corresponding scaling diagram.

(d) Explain in detail the various unstable and stable fixed points in your diagram, emphasizing the physics.

2. Consider the symmetric Anderson model, with a symmetric band-structure at half filling, given by

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^\dagger{}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}}(Vc^\dagger{}_{\mathbf{k}\sigma}f_\sigma + \text{H.c}) + \frac{U}{2}(n_f - 1)^2. \quad (3)$$

where the density of states satisfies $\rho(\epsilon) = \rho(-\epsilon)$. In this model, the f^0 and f^2 states are degenerate and there is the possibility of a "charged Kondo effect" when the interaction U is negative. Show that under the "particle-hole" transformation

$$\begin{aligned} c_{k\uparrow} &\rightarrow c_{k\uparrow}, \qquad d_\uparrow \rightarrow d_\uparrow \\ c_{k\downarrow} &\rightarrow -c^\dagger{}_{k\downarrow}, \qquad f_\downarrow \rightarrow f^\dagger{}_\downarrow \end{aligned} \quad (4)$$

the sign of U reverses, so that the positive U model is transformed to the negative U model.

(a) Show that under this transformation, spin operators of the local moment are transformed into Nambu "isospin operators" which describe the charge and pair degrees of freedom of the d-state, i.e

$$\vec{S} = f^{\dagger}\left(\frac{\vec{\sigma}}{2}\right) f \leftrightarrow \vec{\mathcal{T}} = \tilde{f}^{\dagger}\left(\frac{\vec{\tau}}{2}\right)\tilde{f} \tag{5}$$

where $\tilde{f}^{\dagger} = (f^{\dagger}{}_{\uparrow}, f_{\downarrow})$ is the Nambu spinor for the f-electrons and $\vec{\tau}$ is used to denote the Pauli ("Nambu") matrices, acting in particle-hole space.

(b) Use this transformation to prove that when U is negative, a charged Kondo effect will occur at exactly half-filling involving quantum fluctuations between the degenerate d^0 and d^2 configurations.

(c) If the spin susceptibilty is enhanced in the spin Kondo effect, what susceptibilities are enhanced in the charge Kondo effect?

(d) This *negative* U Kondo effect is believed to occur in the doped semiconductor PbTe when doped with Tl. (See the paper by Y. Matsushita et al, Phys. Rev. Lett. 94, 157002 (2005)). If magnetic impurities like to magnetic order, what kind of order is expected for negative U impurities? Answers will only be accepted with an explanation.

3. **Kondo effect in an insulator and an s-wave superconductor.** Use the large N mean field theory to contrast the effect of an insulating and superconducting gap on the Kondo resonance in the single impurity Kondo effect. Take the large N limit for a Kondo resonance lying at the Fermi energy, with $Q/N = \frac{1}{2}$. In a metallic environment, the mean-field f-electron (composite fermion) spectral function is a simple Lorentzian

$$A_f = \frac{1}{\pi}\mathrm{Im}G_f(\omega - i\delta) = \frac{1}{\pi}\mathrm{Im}[\frac{1}{\omega - i\Delta}] = \frac{1}{\pi}\left(\frac{\Delta}{\omega^2 + \Delta^2}\right) \tag{6}$$

where $\Delta = \pi\rho|V|^2 \sim T_K$ is the resonant level width. Once a gap develops at the Fermi surface, provided the Kondo temperature is large enough compared with the gap, a Kondo effect can still take place and the f-propagator is modified to the form

$$G_f(\omega - i\delta) = \frac{1}{\omega - \Sigma(\omega)} \tag{7}$$

(a) Assuming a constant density of states with an insulating gap of gap Δ_g, show that in the insulator

$$\Sigma_I(\omega - i\delta) = i\Delta + \frac{\Delta}{\pi}\ln\left[\frac{\omega - \Delta_g - i\delta}{\omega + \Delta_g - i\delta}\right] \tag{8}$$

Plot the spectral function of the f-electron.

(b) Show that in an s-wave superconductor with gap function Δ_g, the self-energy becomes

$$\Sigma_{sc}(\omega - i\delta) = i\Delta \frac{1}{\sqrt{(\omega - i\delta)^2 - \Delta_g^2}}(\omega \underline{1} + \Delta_g \tau_1) \tag{9}$$

where τ_1 is the first Nambu matrix. Plot this spectral function and contrast it with the result obtained in the insulator. What is the meaning of the anomalous component of the f-spectral function?

(c) Discuss the main physical difference between the Kondo effect in an insulator and the Kondo effect in an s-wave superconductor.