682A. Solutions to Exercises II. April 2015

1. *Carry out a Hubbard Stratonovich transformation for the following interaction Hamiltonians, factorizing the terms in brackets, and write one or two sentences interpreting the effective Hamiltonian that it gives rise to*

   (a)

$$\begin{aligned} H_I &= -g\int d^3x(\psi^\dagger{}_\uparrow(x)\psi^\dagger{}_\downarrow(x))(\psi_\downarrow(x)\psi_\uparrow(x)) \\ &\rightarrow \int d^3x\left[\bar{\Delta}(x)(\psi_\downarrow(x)\psi_\uparrow(x)) + (\psi^\dagger{}_\uparrow(x)\psi^\dagger{}_\downarrow(x))\Delta(x) + \frac{\bar{\Delta}(x)\Delta(x)}{g}\right]. \end{aligned} \tag{1}$$

   This is the "local" version of the BCS Hamiltonian introduced by Gorkov. It is useful if one wants to consider non-uniform (such as a phase-twisted) solutions.

   (b)

$$\begin{aligned} H_I &= -\frac{J_H}{2}\sum_{(i,j),\sigma,\sigma'}\left(f^\dagger{}_{i\sigma}f_{j\sigma}\right)\left(f^\dagger{}_{j\sigma'}f_{i\sigma'}\right) \\ &\rightarrow \sum_{(i,j)\sigma}\left[\left(f^\dagger{}_{i\sigma}f_{j\sigma}\right)\Delta_{ij} + \bar{\Delta}_{ij}\left(f^\dagger{}_{j\sigma}f_{i\sigma}\right) + 2\frac{\bar{\Delta}_{ij}\Delta_{ij}}{J_H}\right]. \end{aligned} \tag{2}$$

   This mean-field decoupling is often used to describe spin-liquids. The original Hamiltonian does not change the charge $n_f(i)$ at each site. The decoupled Hamiltonian develops a gauge symmetry $f_{j\sigma} \rightarrow e^{i\phi_j}f_{j\sigma}$, $\Delta_{ij} \rightarrow e^{i(\phi_i-\phi_j)}\Delta_{ij}$ which protects the charge conservation at each site.

   (c)

$$\begin{aligned} H_I &= -\frac{J}{2}\sum_{\alpha,\beta}\left(\psi^\dagger{}_\alpha b_\alpha\right)\left(b^\dagger{}_\beta\psi_\beta\right) \\ &\rightarrow \sum_\sigma\left[\left(\psi^\dagger{}_\sigma b_\sigma\right)\alpha + \bar{\alpha}\left(b^\dagger{}_\sigma\psi_\sigma\right) + 2\frac{\bar{\alpha}\alpha}{J}\right] \end{aligned} \tag{3}$$

   This is an unusual Hubbard Stratonovich transformation, because the decoupling field is a Grassman. This field never condenses, so that the above interaction has to be treated beyond mean-field theory at the level of Gaussian fluctuations.

2. *The interaction between an electron gas and the potential field is given by*

$$S_{EM} = \int_0^\beta d\tau\int d^3x\left[e\rho(x)\phi(x) - \frac{\epsilon_0(\nabla\phi)^2}{2}\right] \tag{4}$$

   (My apologies - there was a sign error infront of the $e\rho(x)$.)

(a) *Write down the path integral for a Coulomb gas of quantum electrons, involving an integral over the electron and potential fields.*

The path integral takes the form

$$\begin{aligned} Z &= \int \mathcal{D}[\bar{\psi}, \psi, \phi] e^{-S} \\ S &= \int d\tau d^3x \left[\bar{\psi}(x) \left(\partial_\tau - \frac{\nabla^2}{2m} + e\phi(x) - \mu \right) \psi(x) - \epsilon_0 \frac{(\nabla\phi)^2}{2} \right]. \end{aligned} \tag{5}$$

(b) *What equation does the potential ϕ satisfy at the saddle point of this path integral?*

The effective action is determined by the equation

$$e^{-S_{eff}[\phi]} = \int \mathcal{D}[\bar{\psi}, \psi] e^{-S[\bar{\psi},\psi,\phi]} \tag{6}$$

We can obtain the equation of motion for ϕ by differentiating this expression with respect to ϕ, as follows:

$$\frac{\delta S_{eff}}{\delta\phi(x)} = -\frac{\delta \ln Z}{\delta\phi(x)} = \frac{1}{Z} \int \mathcal{D}[\bar{\psi}, \psi] \left[e\rho(x) + \epsilon_0 \nabla^2 \phi(x) \right] e^{-S} = 0 \tag{7}$$

so that the saddle point solution satisfies Gauss' law

$$-\nabla^2 \phi = \frac{e\rho(x)}{\epsilon_0} \tag{8}$$

or $\nabla \cdot \vec{E} = e\rho/\epsilon_0$.

(c) *Why is the coefficient of the $(\nabla\phi)^2$ term negative? (Give a physical interpretation).*

At first sight, this seems strange, because the last term in the action is minus one times the electromagnetic field energy: i.e minus one times the Coulomb energy! When we write the potential energy as $e\rho\phi$ we overcount the Coulomb energy by a factor of two. The negative correction term not only establishes Gauss's law as the saddle point, it also corrects for the overcounting.

If you eliminate the particle density in terms of the potential, what is the energy density associated with the resulting electric field $\vec{E} = -\vec{\nabla}\phi$?

The answer to this question helps to explain the explanation just given above. If we insert the solution to Gauss's law into the total energy, replacing $e\rho(x) = -\epsilon_0 \nabla^2 \phi$, we obtain

$$\begin{aligned} S_{EM} &= \int d^3x \left[\overbrace{-\epsilon_0 \phi \nabla^2 \phi}^{\to \epsilon_0 (\nabla\phi)^2} - \epsilon_o \frac{(\nabla\phi)^2}{2} \right] \\ &= \int d^3x \left[\epsilon_o \frac{(\nabla\phi)^2}{2} \right] \end{aligned} \tag{9}$$

which is precisely the classical electric field energy, or Coulomb energy.

(d) *Write down the effective action of the system when the fermions have been integrated out and interpret your result in terms of Feynman diagrams.*

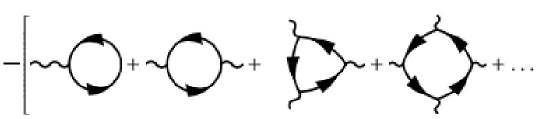
The effective action is given by

$$F_{eff} = -T \ln \mathrm{Det}\left(\overbrace{\partial_\tau - \frac{\nabla^2}{2m} - \mu}^{-G^{-1}} + e\phi \right) - \frac{1}{\beta}\int d^3x d\tau \epsilon_0 \frac{(\nabla\phi)^2}{2} \tag{10}$$

We can expand the logarithm as a power-series in the scattering potential as

$$\begin{aligned} F &= -T\mathrm{Tr}\ln[-G^{-1} + e\phi] - \frac{1}{\beta}\int d^3x d\tau \epsilon_0 \frac{E^2}{2} \\ &= -T\mathrm{Tr}\ln[-G^{-1}] - T\mathrm{Tr}\ln[1 - Ge\phi] - \frac{1}{\beta}\int d^3x d\tau \epsilon_0 \frac{E^2}{2} \\ &= -T\mathrm{Tr}\ln[-G^{-1}] - \frac{1}{\beta}\int d^3x d\tau \epsilon_0 \frac{E^2}{2} + \sum_n \frac{1}{n} T\mathrm{Tr}\left[(Ge\phi)^n\right] \end{aligned} \tag{11}$$

We can interpret the last term as the sum of loop diagrams for repeated scattering off the potential field $\phi(x)$.

$-T\mathrm{Tr}\ln[1 - G(e\phi)] = -$ 

(e) *Suppose the potential field acquires a "mass" term:*

$$S_{EM} = \int_0^\beta d\tau \int d^3x \left[-e\rho(x)\phi(x) - \epsilon_0 \frac{\phi(-\nabla^2 + \kappa^2)\phi}{2} \right] \tag{12}$$

What form does S_{EM} take (in real space) when one integrates out the potential field?

The electromagnetic part of the action can be written in the short-hand

$$S_{EM} = e\rho \cdot \phi + \frac{\phi \cdot M \cdot \phi}{2}$$

where $M \equiv \epsilon_0(\nabla^2 - \kappa^2) \equiv \epsilon_0(-q^2 - \kappa^2)$ in Fourier space. If we carry out the Gaussian integral over ϕ this gives

$$S_{EM} \to -\frac{1}{2} e\rho \cdot M^{-1} \cdot e\rho$$

or written out more explicitly

$$S_{EM} = \frac{1}{2}\int d\tau d^3x d^3x' e^2 \rho(x) \left[\frac{1}{\epsilon_0(\kappa^2 - \nabla^2)} \right]_{x-x'} \rho(x') \tag{13}$$

Now

$$\frac{1}{\epsilon_0(\kappa^2-\nabla^2)} \to \frac{1}{\epsilon_0(q^2+\kappa^2)} \to \frac{1}{4\pi\epsilon_0}\frac{e^{-\kappa r}}{r}$$

so the final action corresponds to fermions interacting with a Yukawa potential.

$$S = \int d\tau \left[\int d^3x\, \psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu\right)\psi + \frac{1}{2}\int d^3x d^3x' V(x-x') : \rho(x)\rho(x') : \right] \tag{14}$$

where

$$V(x-x') = \frac{1}{4\pi\epsilon_0}\frac{e^{-\kappa|x-x'|}}{|x-x'|}.$$

3. *Consider the Euclidean action for a single Harmonic oscillator*

$$S_E = \int_0^\beta d\tau \left[\bar{a}\left(\partial_\tau + \omega\right) a\right]$$

(a) *By re-writing the fields a and $\bar{a}$ in terms of their momentum and position co-ordinates derive the action in terms of p and x. (Hint: terms like $\int d\tau x \partial_\tau x = \frac{1}{2}\int d\tau \partial_\tau (x^2) = 0$ vanish because they are perfect differentials of functions periodic in β).*

If we substitute

$$\begin{aligned} a &= \sqrt{\frac{\omega}{2}}(x + i\frac{p}{\omega}), \\ \bar{a} &= \sqrt{\frac{\omega}{2}}(x - i\frac{p}{\omega}), \end{aligned} \tag{15}$$

then,

$$\begin{aligned} \int_0^\beta \bar{a}(\partial_\tau + \omega)a &= \frac{\omega}{2}\int_0^\beta (x - i\frac{p}{\omega})(\partial_\tau + \omega)(x + i\frac{p}{\omega}) \\ &= \frac{1}{2}\int_0^\beta \left[\omega x\dot{x} + \frac{1}{\omega}p\dot{p} - i(p\dot{x} - x\dot{p}) + \omega^2x^2 + p^2\right] \\ &= \overbrace{\left[\frac{\omega}{4}x^2 + \frac{1}{4\omega}p^2 + \frac{i}{2}xp\right]_0^\beta}^{=0} + \frac{1}{2}\int_0^\beta d\tau \left[-ip\dot{x} + H(p,x)\right] \\ &= \frac{1}{2}\int_0^\beta d\tau \left[-ip\dot{x} + H(p,x)\right] \end{aligned} \tag{16}$$

where the exact integrand appearing on line 3 is eliminated due to periodic boundary conditions, and $H(p,x) = \frac{p^2}{2} + \frac{\omega^2}{2}x^2$ is the Hamiltonian

(b) *Carry out the Gaussian integral over the momentum field p and derive the corresponding action in terms of $\dot{x}$ and x.*

If we integrate the action over momentum, then

$$\int d\tau p(-i\dot{x}) + \frac{p^2}{2} \rightarrow -\int d\tau (-i\dot{x})^2 = \int d\tau \frac{\dot{x}^2}{2} \tag{17}$$

so the final action is

$$S = \int_0^\beta d\tau \left(\frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2} \right). \tag{18}$$

4. **Mean field theory for antiferromagnetic Spin Density Wave** Develop the mean-field theory for a three dimensional tight-binding cubic lattice with commensurate antiferromagnetic order parameter

$$\mathbf{M}_j = \mathbf{M} e^{i\mathbf{Q}\cdot\mathbf{R}_j} \tag{19}$$

where $\mathbf{Q} = (\pi, \pi, \pi)$.

(a) *Show that the mean-field free energy can be written in the form*

$$H_{MF} = \sum_{\mathbf{k}\in\frac{1}{2}BZ} \psi^{\dagger}{}_{\mathbf{k}} \begin{pmatrix} \epsilon_{\mathbf{k}} - \mu & \mathbf{M}\cdot\vec{\sigma} \\ \mathbf{M}\cdot\vec{\sigma} & \epsilon_{\mathbf{k}+\mathbf{Q}} - \mu \end{pmatrix} \psi_{\mathbf{k}} + \mathcal{N}_s \frac{M^2}{2I} \tag{20}$$

where $M = |\mathbf{M}|$ is the magnitude of the staggered mangetization, $\psi_{\mathbf{k}}$ denotes the four-component spinor

$$\psi_{\mathbf{k}} = \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{\mathbf{k}\downarrow} \\ c_{\mathbf{k}+\mathbf{Q}\uparrow} \\ c_{\mathbf{k}+\mathbf{Q}\downarrow} \end{pmatrix}, \tag{21}$$

$\epsilon_{\mathbf{k}} = -2t(c_x + c_y + c_z)$, ($c_l \equiv \cos k_l$, $l = x, y, z$) is the kinetic part of the energy and the summation is restricted to the magnetic Brillouin zone, (One half the original Brillouin zone.)

The action for the electrons moving in the Weiss field is given by

$$S = \int d\tau \left[\sum_{\mathbf{k}} c^{\dagger}{}_{\mathbf{k}} \left(\partial_\tau + \epsilon_{\mathbf{k}} - \mu \right) c_{\mathbf{k}} \right] + \sum_j \left(\vec{M}_j \cdot (c^{\dagger}{}_j \vec{\sigma} c_j) + \frac{\vec{M}_j^2}{2I} \right) \tag{22}$$

from which we can read off the mean-field Hamiltonian

$$H_{MF} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) c^{\dagger}{}_{\mathbf{k}} c_{\mathbf{k}} + \sum_j \left(\vec{M}_j \cdot (c^{\dagger}{}_j \vec{\sigma} c_j) + \frac{\vec{M}_j^2}{2I} \right) \tag{23}$$

If we now substitute

$$\vec{M}_j = \vec{M} e^{i\vec{Q}\cdot\vec{R}_j} \tag{24}$$

where $\vec{Q} = (\pi, \pi, \pi)$, then

$$H_{MF} = \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) c^{\dagger}{}_{\mathbf{k}} c_{\mathbf{k}} + \sum_{\mathbf{k}} \vec{M} \cdot (c^{\dagger}{}_{\mathbf{k+Q}} \vec{\sigma} c_{\mathbf{k}}) + \mathcal{N}_s \frac{\vec{M}^2}{2I}. \tag{25}$$

Dividing the Brillouin zone into two halves, we obtain

$$\begin{aligned} H_{MF} &= \sum_{\mathbf{k}\in\frac{1}{2}BZ} \left[(\epsilon_k - \mu) c^{\dagger}{}_{\mathbf{k}} c_{\mathbf{k}} + (\epsilon_{\mathbf{k+Q}} - \mu) c^{\dagger}{}_{\mathbf{k+Q}} c_{\mathbf{k+Q}} + \left(\vec{M} \cdot (c^{\dagger}{}_{\mathbf{k+Q}} \vec{\sigma} c_{\mathbf{k}}) + \text{H.c} \right) \right] + \mathcal{N}_s \frac{\vec{M}^2}{2I} \\ &= \sum_{\mathbf{k}\in\frac{1}{2}BZ} \psi^{\dagger}{}_{\mathbf{k}} \begin{pmatrix} \epsilon_{\mathbf{k}} - \mu & \mathbf{M}\cdot\vec{\sigma} \\ \mathbf{M}\cdot\vec{\sigma} & \epsilon_{\mathbf{k+Q}} - \mu \end{pmatrix} \psi_{\mathbf{k}} + \mathcal{N}_s \frac{M^2}{2I}. \end{aligned} \tag{26}$$

(b) *On a tight binding lattice the kinetic energy has the "nesting" property that $\epsilon_{\mathbf{k+Q}} = -\epsilon_{\mathbf{k}}$. Show that the energy eigenvalues of the mean-field Hamiltonian have a BCS form*

$$E_{\mathbf{k}\pm} = \pm\sqrt{\epsilon_{\mathbf{k}}^2 + M^2} - \mu. \tag{27}$$

For a nested Fermi surface, the mean-field Hamiltonian becomes

$$H_{MF} = \sum_{\mathbf{k}\in\frac{1}{2}BZ} \psi^{\dagger}{}_{\mathbf{k}} h(\mathbf{k}) \psi_{\mathbf{k}} + \mathcal{N}_s \frac{M^2}{2I}. \tag{28}$$

where the matrix hamiltonian

$$h(\mathbf{k}) = \begin{pmatrix} \epsilon_{\mathbf{k}} - \mu & \mathbf{M}\cdot\vec{\sigma} \\ \mathbf{M}\cdot\vec{\sigma} & -\epsilon_{\mathbf{k}} - \mu \end{pmatrix} = \epsilon_{\mathbf{k}}\tau_3 + \vec{M}\cdot\vec{\sigma}\tau_1 - \mu\underline{1} \tag{29}$$

where the four dimensional Nambu matrices are

$$\tau_3 = \begin{pmatrix} \underline{1} & 0 \\ 0 & -\underline{1} \end{pmatrix}, \qquad \vec{\sigma}\tau_1 = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}. \tag{30}$$

The quasiparticle energies are determined by the eigenvalues of $h(\mathbf{k})$. We can determine these from

$$\text{Det}[E - (\epsilon_{\mathbf{k}}\tau_3 - \vec{M}\cdot\vec{\sigma}\tau_1 + \mu)] = 0 \tag{31}$$

We can determine the eigenvalues by squaring the determinant, to obtain

$$\text{Det}\left[\left(E - (\epsilon_{\mathbf{k}}\tau_3 - \vec{M}\cdot\vec{\sigma}\tau_1 + \mu) \right) \left(E + (\epsilon_{\mathbf{k}}\tau_3 + \vec{M}\cdot\vec{\sigma}\tau_1 + \mu) \right) \right] = \text{Det}\left[(E+\mu)^2 - (\epsilon_{\mathbf{k}}^2 + \vec{M}^2) \right]$$

(where we have used the fact that τ_3 anticommutes with $\vec{\sigma}\tau_1$). From this, we see that

$$E_{\mathbf{k}p} = -\mu \pm \sqrt{\epsilon_{\mathbf{k}}^2 + \vec{M}^2} \tag{32}$$

Note that in the half Brillouin zone, each of the corresponding eigenstates has a two-fold spin degeneracy.

(c) *Show that the mean-field free energy takes the form*

$$F = \sum_{\mathbf{k},p=\pm 1} -T \ln\left[2\cosh\left(\frac{\beta E_{\mathbf{k}p}}{2}\right)\right] + \mathcal{N}_s\left(\frac{M^2}{2I} - 2\mu\right) \tag{33}$$

From the energy eigenstates, we see that the mean-field Free energy is given by

$$F = -2T \sum_{\mathbf{k}\in\frac{1}{2}BZ,p=\pm} \ln(1 + e^{-\beta E_{\mathbf{k}p}}) + \mathcal{N}_s \frac{\vec{M}^2}{2I} \tag{34}$$

where the factor of two is from the two-fold spin degeneracy of the states. We can expand the summation to the full Brillouin zone by dividing the first term by two,

$$\begin{aligned} F &= -T \sum_{\mathbf{k}\in BZ,p=\pm} \ln(1 + e^{-\beta E_{\mathbf{k}p}}) + \mathcal{N}_s \frac{\vec{M}^2}{2I} \\ &= -T \sum_{\mathbf{k}\in BZ} \ln\left[(1 + e^{-\beta E_{\mathbf{k}+}})(1 + e^{-\beta E_{\mathbf{k}-}})\right] + \mathcal{N}_s \frac{\vec{M}^2}{2I} \end{aligned} \tag{35}$$

Now with a bit of algebra, we deduce that

$$\begin{aligned} (1 + e^{-\beta E_{\mathbf{k}+}})(1 + e^{-\beta E_{\mathbf{k}-}}) &= (1 + e^{-\beta(-\mu+\sqrt{\quad})})(1 + e^{-\beta(-\mu-\sqrt{\quad})}) \\ &= e^{-\frac{\beta}{2}(-\mu+\sqrt{\quad})} e^{-\frac{\beta}{2}(-\mu-\sqrt{\quad})} 4\cosh\left(\frac{\beta E_{\mathbf{k}+}}{2}\right)\cosh\left(\frac{\beta E_{\mathbf{k}-}}{2}\right) \\ &= e^{\beta\mu} 4\cosh\left(\frac{\beta E_{\mathbf{k}+}}{2}\right)\cosh\left(\frac{\beta E_{\mathbf{k}-}}{2}\right) \end{aligned} \tag{36}$$

so we can write the mean field Free energy in the form

$$\begin{aligned} F &= \sum_{\mathbf{k},p=\pm} \left[-T \ln\cosh\left(\frac{\beta E_{\mathbf{k}p}}{2}\right) - \frac{\mu}{2}\right] + \mathcal{N}_s \frac{\vec{M}^2}{2I} \\ &= \sum_{\mathbf{k},p=\pm 1} -T \ln\left[2\cosh\left(\frac{\beta E_{\mathbf{k}p}}{2}\right)\right] + \mathcal{N}_s\left(\frac{M^2}{2I} - \mu\right) \end{aligned} \tag{37}$$

Note the factor of two difference with the question - a typo in the original form. We can check that the above form is the correct one, by differentiating w.r.t μ to get the total particle number

$$\begin{aligned} N &= -\frac{\partial F}{\partial \mu} = \sum_{\mathbf{k},p=\pm} = -\frac{1}{2}\tanh\left(\frac{\beta E_{\mathbf{k}p}}{2}\right) + \mathcal{N}_s \\ &= \sum_{\mathbf{k},p=\pm} \frac{1}{2}\left(1 - \tanh\left(\frac{\beta E_{\mathbf{k}p}}{2}\right)\right) = \sum_{\mathbf{k},p=\pm} f(E_{\mathbf{k}p}) \end{aligned} \tag{38}$$

(d) Differentiating the mean-field Free energy w.r.t $\vec{M}$, we obtain

$$\frac{1}{\mathcal{N}_s}\frac{\partial F}{\partial \vec{M}} = -\int_{\mathbf{k}} \tanh\left(\frac{\beta E_{\mathbf{k}p}}{2}\right) \frac{\vec{M}p}{2\sqrt{\epsilon_{\mathbf{k}}^2 + M^2}} + \frac{\vec{M}}{I} = 0 \tag{39}$$

from which we obtain

$$\frac{1}{2}\sum_{\mathbf{k},p=\pm 1} \tanh\left(\frac{\sqrt{\epsilon_{\mathbf{k}}^2 + M^2} - \mu p}{2T}\right) \frac{1}{\sqrt{\epsilon_{\mathbf{k}}^2 + M^2}} = \frac{1}{I}. \tag{40}$$

(e) *Show that at half filling, the nesting guarantees that a transition to a spin-density wave will occur for arbitrarily small interaction strength I.*

The gap equation at $T = 0$ is given by

$$\sum_{\mathbf{k},p} \frac{\mathrm{sgn}(pE_{\mathbf{k}p})}{2\sqrt{\epsilon_{\mathbf{k}}^2 + M^2}} = \frac{1}{I}. \tag{41}$$

If we set $M = 0$, we obtain the critical $I = I_c$ for an instability into the antiferromagnet,

$$\begin{aligned} \frac{1}{I_c} &= \sum_{\mathbf{k},p} \frac{\mathrm{sgn}(|\epsilon_{\mathbf{k}}| - p\mu)}{2|\epsilon_{\mathbf{k}}|} \\ &= \sum_p \int_{-\Lambda}^{\Lambda} d\epsilon N(\epsilon) \frac{\mathrm{sgn}(|\epsilon| - p\mu)}{2|\epsilon|} \\ &= \sum_p \int_0^{\Lambda} d\epsilon N(\epsilon) \frac{\mathrm{sgn}(|\epsilon| - p\mu)}{|\epsilon|} \\ &= 2\int_{\mu}^{\Lambda} \frac{d\epsilon}{\epsilon} \\ &\sim 2N(0)\ln\left(\frac{\Lambda}{\mu}\right) \end{aligned} \tag{42}$$

where we have replaced the integral over **k** by an integral over the density of states, introducing a cutoff Λ. To extract the infra-red divergence in the integral, we have replaced the density of states per spin by its value at $\epsilon = 0$, $N(0)$. The critical value I_c is then given by

$$2I_cN(0) \sim \frac{1}{\ln\left(\frac{\Lambda}{\mu}\right)} \tag{43}$$

a quantity that goes to zero as $\mu \to 0$. In other words, the critical value of I_c is zero at particle-hole symmetry.

(f) *Calculate the [zero temperature] phase diagram assuming that the order remains commensurate at finite doping.*

A precise answer to this question would plot out the T_c versus doping, determined from the two equations

$$1 + \delta = \sum_{p=\pm} \int \frac{d^3k}{(2\pi)^3} f(E_{\mathbf{k},p}),$$

where δ is the doping away from half filling and

$$\sum_{p=\pm1} \int \frac{d^3k}{(2\pi)^3} \tanh\left(\frac{\sqrt{\epsilon_{\mathbf{k}}^2 + M^2} - \mu p}{2T}\right) \frac{1}{2\sqrt{\epsilon_{\mathbf{k}}^2 + M^2}} = \frac{1}{I}.$$

However, lets do it heuristically for the moment. Lets look first at the zero temperature phase diagram, plotting I_c as a function of doping away from half-filling. If the doping is $\delta = 2\mu N(0)$, then we have

$$2I_cN(0) \sim \frac{1}{\ln\left(\frac{2\Lambda N(0)}{\delta}\right)}$$

Putting $2N(0) \sim 1/\epsilon_F$, where ϵ_F is the Fermi energy, we obtain

$$I_c \sim \frac{\epsilon_F}{\ln\left(\frac{(\Lambda/\epsilon_F)}{\delta}\right)}.$$

We can estimate T_c by writing

$$2I_cN(0) \sim \frac{1}{\ln\left(\frac{\Lambda}{T_c}\right)}$$

or

$$T_c \sim \Lambda e^{-1/2I_cN(0)}$$

Its probably just as good to sketch the phase diagram. However, as an exercise I worked out the phase boundary in mathematica, as shown below. Notice that my finite grid size in integration has eliminated the region near $I = 0$ and $\delta = 0$. With some work this could have been fixed, but I drew the lines in by hand.

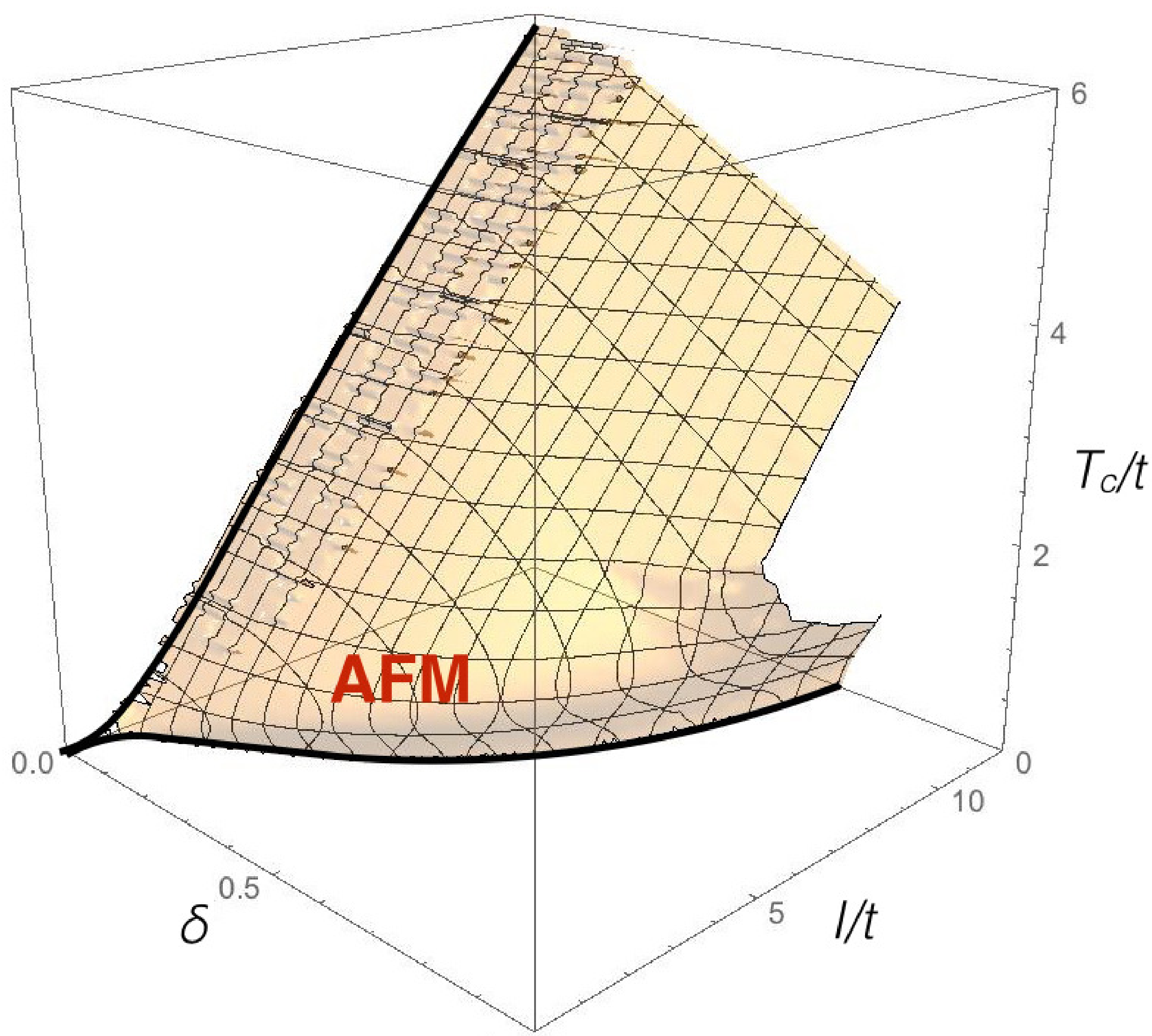


Figure 1: Phase boundary for antiferromagnet as a function of doping, calculated in mean field theory. All units with $t = 1$.