682A. Solutions to Exercises 1. March 2015

1. (a) Using a Taylor expansion, and noting that $\alpha^n = 0$ for all $n > 1$,

$$(1+\alpha)^{-1} = (1 - \alpha + (-\alpha)^2 + \dots) = 1 - \alpha.$$

We can't invert $1/\alpha$ because there is no Taylor expansion around $\alpha = 0$. Lets try to calculate the inverse of alpha by looking at

$$(\epsilon + \alpha)^{-1} = \frac{1}{\epsilon(1 + (\frac{\alpha}{\epsilon}))} = \frac{1}{\epsilon} - \alpha.$$

Unfortunately, we can't take the limit $\epsilon \to 0$, due to the singular term. There is thus no Grassman number corresponding to α^{-1}.

(b) Using a binomial expansion, noting that the expansion truncates at linear order, we obtain:

$$\sqrt{(1+\bar{\alpha}\alpha)} = \left(1 + \frac{1}{2}\bar{\alpha}\alpha - \frac{1}{8}(\bar{\alpha}\alpha)^2 + \dots\right) = 1 + \frac{\bar{\alpha}\alpha}{2}.$$

(c) Carrying out the Taylor expansion, in this case, the series truncates at second order:

$$\cos[\bar{\alpha}\alpha + \bar{\beta}\beta] = 1 - \frac{1}{2}(\bar{\alpha}\alpha + \bar{\beta}\beta)^2 + \frac{1}{6}(\bar{\alpha}\alpha + \bar{\beta}\beta)^4 = 1 - (\bar{\alpha}\alpha)(\bar{\beta}\beta). \tag{1}$$

(d) Similarly, carrying out a Taylor expansion of the matrix exponential,

$$\exp\begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix} = 1 + \begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 - \frac{1}{2}\bar{\alpha}\alpha & \alpha \\ \bar{\alpha} & 1 + \frac{1}{2}\bar{\alpha}\alpha \end{pmatrix}.$$

(e) If $f(\bar{\alpha}, \alpha) = f + \bar{\beta}\alpha - \bar{\alpha}\beta + g\bar{\alpha}\alpha$ is a Grassmanian function of two variables, then

i.
$$\frac{\partial f(\bar{\alpha}, \alpha)}{\partial \alpha} = -\bar{\beta} - g\bar{\alpha},$$

ii.
$$\frac{\partial^2 f(\bar{\alpha}, \alpha)}{\partial \bar{\alpha} \partial \alpha} = \frac{\partial}{\partial \bar{\alpha}}(-\bar{\beta} - g\bar{\alpha}) = -g,$$

iii. Since differentiation over Grassman variables is equivalent to integration

$$\int d\bar{\alpha} d\alpha f(\bar{\alpha}, \alpha) = \frac{\partial^2 f(\bar{\alpha}, \alpha)}{\partial \bar{\alpha} \partial \alpha} = -g.$$

(f) $\int d\bar{\alpha}_1 d\alpha_1 d\bar{\alpha}_2 d\alpha_2 \exp\left[\begin{pmatrix}\bar{\alpha}_1, \bar{\alpha}_2\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right] = \det\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} = -5.$

2. In this question, I wanted to check that you had understood how to set up a fermionic path integral.

(a) The first step in setting up the path integral is to write the Trace using coherent states. For a general operator $\hat{A}$, we may write

$$\begin{aligned}\langle \bar{c}|\hat{A}|c\rangle &= A_{oo}\langle \bar{c}|0\rangle\langle 0|c\rangle + A_{01}\langle \bar{c}|0\rangle\langle 1|c\rangle + A_{10}\langle \bar{c}|1\rangle\langle 0|c\rangle + A_{11}\langle \bar{c}|1\rangle\langle 1|c\rangle \\ &= A_{oo} + A_{01}c + A_{10}\bar{c} + A_{11}\bar{c}c, \end{aligned} \tag{2}$$

so that the trace may be written

$$Tr[A] = A_{00} + A_{11} = -\int d\bar{c}dc e^{\bar{c}c}\langle \bar{c}|\hat{A}|c\rangle. \tag{3}$$

Applying this to the partition function,

$$\begin{aligned}Tr[e^{-\beta H}] &= -\int d\bar{c}_3 dc_0 e^{\bar{c}_3 c_0}\langle \bar{c}_3|e^{-\beta H}|c_0\rangle \\ &= \int d\bar{c}_3 dc_3 e^{-\bar{c}_3 c_3}\langle \bar{c}_3|e^{-\beta H}|c_0\rangle, \end{aligned} \tag{4}$$

where we have used the definition, $c_3 = -c_0$. We now use the completeness relation

$$1 = \int d\bar{c}dc e^{-\bar{c}c}|c\rangle\langle c| \tag{5}$$

to introduce two time-slices into the matrix element $\langle \bar{c}_3|e^{-\beta H}|c_3\rangle$, which we write as

$$\langle \bar{c}_3|e^{-\beta H}|c_3\rangle = \int d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \langle \bar{c}_3|e^{-\Delta\tau H}|c_2\rangle\langle \bar{c}_2|e^{-\Delta\tau H}|c_1\rangle\langle \bar{c}_1|e^{-\Delta\tau H}|c_0\rangle e^{-\bar{c}_1 c_1 - \bar{c}_2 c_2}. \tag{6}$$

Finally, using the expansion of the matrix element in terms of coherent states,

$$\langle \bar{c}_{j+1}|e^{-\Delta\tau H}|c_j\rangle = e^{\alpha\bar{c}_{j+1}c_j} + O(\Delta\tau^2), \tag{7}$$

where $\alpha = (1 - \Delta\tau\epsilon)$, we obtain

$$\begin{aligned} Z_3 &= \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 e^{\alpha[\bar{c}_3 c_2 + \bar{c}_2 c_1 + \bar{c}_1 c_0] - [\bar{c}_3 c_3 + \bar{c}_2 c_2 + \bar{c}_1 c_1]} \\ &= \int d\bar{c}_3 dc_3 d\bar{c}_2 dc_2 d\bar{c}_1 dc_1 \exp\left\{-(\bar{c}_3, \bar{c}_2, \bar{c}_1)\begin{pmatrix}1 & -\alpha & 0\\ 0 & 1 & -\alpha\\ \alpha & 0 & 1\end{pmatrix}\begin{pmatrix}c_3\\ c_2\\ c_1\end{pmatrix}\right\}, \end{aligned} \tag{8}$$

where we have set $c_0 = -c_3$ in the last step.

(b) Since this integral is Gaussian, the integral is given by the determinant of the matrix:

$$Z_3 = \det\begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ \alpha & 0 & 1 \end{pmatrix} = 1 + \alpha^3. \tag{9}$$

(c) Generalizing this result to N time-slices, we obtain

$$\begin{aligned} Z_N &= \det[\mathcal{M}] \\ \mathcal{M} &= \begin{pmatrix} 1 & -\alpha & 0 & \dots & 0 \\ 0 & 1 & -\alpha & \dots & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & -\alpha \\ \alpha & \dots & \dots & \dots & 1 \end{pmatrix} \\ \det[\mathcal{M}] &= 1 + \alpha^N \qquad \text{(by inspection).} \end{aligned} \tag{10}$$

In the limit $N \to \infty$,

$$Lt_{N\to\infty}\left(1 - \frac{\beta\epsilon}{N}\right)^N = e^{-\beta\epsilon}, \tag{11}$$

so that

$$Z_N \to 1 + e^{-\beta\epsilon}. \tag{12}$$

3. We need to evaluate

$$Z = \int \mathcal{D}[\bar{f}, f] e^{-S}, \tag{13}$$

where

$$S = \int d\tau \bar{f}_\alpha \left(\partial_\tau - \vec{\sigma}\cdot\vec{B}\right)_{\alpha,\beta} f_\beta, \tag{14}$$

where $\vec{B} = B\hat{z}$ is the applied field. Since this is a Gaussian integral,

$$\begin{aligned} Z &= \det\left(\partial_\tau - \vec{\sigma}\cdot\vec{B}\right) \\ F &= -T\ln Z = -T\,\mathrm{Tr}\ln\left(\partial_\tau - \vec{\sigma}\cdot\vec{B}\right). \end{aligned} \tag{15}$$

To evaluate the trace, we go into the frequency domain, $\partial_\tau \to -i\omega_n$

$$F = -T\sum_{i\omega_n} \mathrm{Tr}\ln\left(-i\omega_n - \vec{\sigma}\cdot\vec{B}\right). \tag{16}$$

Formally, to carry out the Matsubara sum, we convert the summation to a contour integral around the poles of the Fermi function $f(z)$. We then distort the integral around the branch-cuts in the logarithm and carry out the integral to obtain:

$$\begin{aligned} F &= -T\sum_{i\omega_n,\sigma} \ln(-i\omega_n - \sigma B) \\ &= \sum_\sigma \oint_{\text{branch-cut}} \frac{dz}{2\pi i} f(z)\ln(-z-\sigma B) \\ &= \sum_\sigma \int_{-\sigma B}^{\infty} \frac{d\omega}{\pi} f(\omega)\text{Im}\ln(-\omega-\sigma B+i\delta) \\ &= \sum_\sigma \int_{-\sigma B}^{\infty} d\omega f(\omega) \\ &= \sum_\sigma -T\ln(1+e^{\beta\sigma B}) \\ &= -2T\ln[2\cosh\frac{\beta B}{2}]. \end{aligned} \quad (17)$$

The partition function is

$$Z = \left(2\cosh\frac{\beta B}{2}\right)^2 = 2\cosh\beta B + 2 = Z_{spin} + 2. \quad (18)$$

Notice how this differs from the partition function of a spin because the remainder term derived from the the empty and doubly occupied states. One way of removing these states is using the "Popov Fedatov" trick, in which one adds an imaginary chemical potential to the Hamiltonian as follows:

$$H = -f^\dagger\left(\vec{\sigma}\cdot\vec{B}\right)f + i\left(\frac{\pi T}{2}\right)(f^\dagger f - 1).$$

The additional complex chemical potential term has the effect of cancelling out the unwanted empty state with the unwanted doubly occupied state, to give the correct partition function.

4. Let us evaluate

$$I = \int d\bar{c}dc\langle -c|\hat{A}|c\rangle e^{-\bar{c}c}, \quad (19)$$

where

$$\langle -\bar{c}|\hat{A}|c\rangle = (A_0 - A_1\bar{c}c)e^{-\bar{c}c}. \quad (20)$$

Carrying out the integral, we have

$$I = \int d\bar{c}dc(A_0 - A_1\bar{c}c)e^{-2\bar{c}c}$$

$$
\begin{aligned}
&= \int d\bar{c}dc(A_0 - A_1\bar{c}c)(1 - 2\bar{c}c) \\
&= \int d\bar{c}dc(-2\bar{c}cA_0 - A_1\bar{c}c) \\
&= (2A_0 + A_1),
\end{aligned}
\tag{21}
$$

as expected.

5. I'm sorry, this was a lot more challenging that I had in mind at first. The hint was only half-way helpful, because I had made a mistake in my original solution. We had

$$
\mathcal{M} = e^{\frac{1}{2}\sum_{i,j} A_{ij} c^{\dagger}{}_i c^{\dagger}{}_j},
$$

where A_{ij} is an $N \times N$ antisymmetric matrix, and the $c^{\dagger}{}_j$ are a set of N canonical Fermi creation operators. Our task is to use coherent states to calculate

$$
t = \mathrm{Tr}[\mathcal{M}\mathcal{M}^{\dagger}],
$$

where the trace is over the 2^N dimensional Hilbert space of fermions. As we shall see, the answer for A real, is simply $t = \det[2 + A]$.

Converting this into Grassmanian calculus, we have

$$
t = \int \prod_i d\bar{c}_i dc_i \langle -\bar{c}|\mathcal{M}\mathcal{M}^{\dagger}|c\rangle e^{-\sum_i \bar{c}_i c_i}. \tag{22}
$$

Now since $\mathcal{M}\mathcal{M}^{\dagger}$ is already normal ordered, we know that

$$
\langle -\bar{c}|\mathcal{M}\mathcal{M}^{\dagger}|c\rangle = \exp\left[\frac{1}{2}\left(\sum_{i,j} A_{ij}\bar{c}_i\bar{c}_j + (\mathrm{H.c})\right)\right] e^{-\sum_i \bar{c}_i c_i},
$$

so that the full trace can be written

$$
\begin{aligned}
t &= \int \prod_i d\bar{c}_i dc_i \exp\left[\frac{1}{2}\left(\sum_{i,j} A_{ij}\bar{c}_i\bar{c}_j + (\mathrm{H.c})\right) - 2\sum_i \bar{c}_i c_i\right] \\
&= \int \prod_i d\bar{c}_i dc_i \exp\left[-\frac{1}{2}\bar{\psi}\begin{pmatrix} \underline{2} & -A \\ -A^{\dagger} & -\underline{2} \end{pmatrix}\psi\right],
\end{aligned}
\tag{23}
$$

where $\underline{2} = 2\underline{1}$ is twice the N dimensional unit matrix and

$$
\bar{\psi} = (\bar{c}_1, \ldots \bar{c}_N, c_1 \ldots c_N), \qquad \psi = \begin{pmatrix} c_1 \\ \vdots \\ c_N \\ \bar{c}_1 \\ \vdots \\ \bar{c}_N \end{pmatrix}, \tag{24}
$$

defines a $2N$ dimensional spinor ψ and its conjugate.

Now (23) is a Gaussian integral, so we can definitely do it, but you've got to be a bit careful, because you can't just take the determinant of the matrix, because the c_i occur in both ψ and $\bar{\psi}$, so ψ and $\bar{\psi}$ aren't independent. You can do the integral in various ways. One way to do it would be to first diagonalize

$$\frac{1}{2}\bar{\psi}\begin{pmatrix} 2\underline{1} & -A \\ -A^\dagger & -2\underline{1} \end{pmatrix}\psi \to \frac{1}{2}\bar{\alpha}\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_N & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_N \end{pmatrix}\alpha = \sum_i \lambda_i \bar{\alpha}_i \alpha_i, \tag{25}$$

where the λ_i are the eigenvalues of the matrix. The doubling of the eigenvalues is guaranteed by the particle-hole symmetry of the matrix. Now when we do the integral over the α_i, we have to notice that the first N and second N terms in the exponential of the integrand are identical and should be grouped together, so that the integral over the α variables is then,

$$t = \int \prod d\bar{\alpha}_i d\alpha_i e^{-\sum_i \lambda_i \bar{\alpha}_i \alpha_i} = \prod_i \lambda_i = \sqrt{(-1)^N \det\begin{pmatrix} \underline{2} & -A \\ -A^\dagger & -\underline{2} \end{pmatrix}}, \tag{26}$$

where the $(-1)^N$ is a result of making the identification

$$\det\begin{pmatrix} \underline{2} & -A \\ -A^\dagger & -\underline{2} \end{pmatrix} = (-1)^N \left(\prod_i \lambda_i\right)^2. \tag{27}$$

Another way to do the integral is to "square it", writing the square as the integral as two separate integrals, then combining them into a single integral in which ψ and $\bar{\psi}$ are independent. (The $(-1)^N$ is picked up in the combination process). The final answer is then the square root of the determinant.

Anyway, we can check our answer for $A = 0$, and this gives 2^N, which is the right answer! You can also check the result for the $N = 2$ case where $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, which gives $t = 4 - a^2$ by direct integration, and from the above formula. We can actually simplify the result a bit more. If we square the matrix, it becomes diagonal, so that

$$\det\begin{pmatrix} \underline{2} & -A \\ -A^\dagger & -\underline{2} \end{pmatrix}^2 = \det\begin{pmatrix} 4 + AA^\dagger & \\ & 4 + A^\dagger A \end{pmatrix} = \det(4 + A^\dagger A)\det(4 + AA^\dagger) = \det(4 + A^\dagger A)^2. \tag{28}$$

In the above expression, we are able to identity $\det(4 + AA^\dagger) = \det(4 + AA^\dagger)^T = \det(4 + A^* A^T) = \det(4 + (-A^\dagger)(-A)) = \det(4 + A^\dagger A)$, where the middle step relied on the antisymmetry of A. Here we've been a bit sloppy, writing 4 where we really should write $\underline{4}$. Thus,

$$\det\begin{pmatrix} \underline{2} & -A \\ -A^\dagger & -\underline{2} \end{pmatrix} = (-1)^N \det(4 + A^\dagger A). \tag{29}$$

Combining everything, we then have

$$t = \mathrm{Tr}[\mathcal{M}\mathcal{M}^\dagger] = \left[\det(4 + A^\dagger A)\right]^{1/2} = [\det(4 - A^* A)]^{1/2}\,, \tag{30}$$

where the last step follows from the antisymmetry of A. For the special case where A is real, so that $A^* = A$, this can be further simplified into

$$t = \sqrt{\det[2 - A]\det[2 + A]} = \sqrt{\det[2 - A]\det[2 - A]^T} = \det[\underline{2} + A]. \tag{31}$$