Extending Luttinger's theorem to Z_2 fractionalized phases of matter

Arun Paramekanti[1] and Ashvin Vishwanath[2]
[1]*Department of Physics and Kavli Institute for Theoretial Physics, University of California, Santa Barbara, California 93106-4030, USA*
[2]*Department of Physics, Massachussetts Institute of Technology, 77 Massachussetts Ave., Cambridge, Massachussetts, 02139 USA*
(Received 24 June 2004; published 23 December 2004)

Luttinger's theorem for Fermi liquids equates the volume enclosed by the Fermi surface in momentum space to the electron filling, independent of the strength and nature of interactions. Motivated by recent momentum balance arguments that establish this result in a nonperturbative fashion [M. Oshikawa, Phys. Rev. Lett. **84**, 3370 (2000)], we present extensions of this momentum balance argument to exotic systems which exhibit quantum number fractionalization focusing on Z_2 fractionalized insulators, superfluids and Fermi liquids. These lead to nontrivial relations between the particle filling and some intrinsic property of these quantum phases, and hence may be regarded as natural extensions of Luttinger's theorem. We find that there is an important distinction between fractionalized states arising naturally from half filling versus those arising from integer filling. We also note how these results can be useful for identifying fractionalized states in numerical experiments.

DOI: 10.1103/PhysRevB.70.245118 PACS number(s): 71.27.+a, 71.10.−w, 75.10.Jm, 75.40.Mg

I. INTRODUCTION

The last two decades have witnessed the experimental discovery of several strongly correlated materials that show properties strikingly different from that expected from conventional theories based on Landau's Fermi liquid picture. These include the high temperature copper oxide superconductors, heavy fermion systems near a quantum critical point, and, more recently, the cobalt oxide materials. Interesting correlated quantum phases are also likely to emerge in the near future from ongoing experimental efforts in the area of cold atoms in optical lattices. It has then become imperative to theoretically investigate quantum phases of matter that differ fundamentally from the standard paradigm. Indeed, in such a search for new theoretical models, it would be useful to know if general principles place constraints on the possible quantum phases. Here, we will explore in detail the consequences of one such constraint, arising from momentum balance, which will be applicable to interacting many body systems on a lattice. This argument was first applied to the case of one-dimensional Luttinger liquids,[1] where it relates the Fermi wave vector k_F to the particle density. It was later extended to Fermi liquids[2] in dimensions $D \geqslant 2$, where it leads to Luttinger's theorem,[3] relating the filling fraction to the volume enclosed within the Fermi surface on which the long lived Fermi-liquid quasiparticles are defined. Here, we will apply the same line of argument to a variety of different phases in spatial dimensions $D > 1$, and the constraint we obtain in this way may be viewed as analogues of Luttinger's theorem for these phases. In all cases, the filling fraction (number of particles per unit cell of the lattice) is fundamentally related to some intrinsic property of the phase.

The momentum balance argument, introduced by Oshikawa for Fermi Liquids,[2] proceeds as follows. Consider a system of interacting fermions at some particular filling on a finite lattice at zero temperature. Periodic boundary conditions implies that the lattice has a torus geometry; imagine introducing a solenoid of flux in one of the holes of the torus, and adiabatically changing its strength from zero to 2π. The crystal momentum imparted to the system can then be calculated in two different ways. First, in a trivial fashion that only depends on the filling and is independent of the quantum phase, the system reaches in the thermodynamic limit, and second, in way that depends essentially on the quantum phase of the system. Consistency requires the equality of the these two quantities—which leads to the nontrivial conditions on the quantum phase. Essentially, each consistent quantum phase has its own way of absorbing the filling dependent crytal momentum that is generated in this process—as mentioned, in the case of the Fermi liquid this leads to Luttinger's theorem.

Here, we begin by applying this argument to the case of (bosonic) insulators at half filling, where the system in the thermodynamic limit necessarily acquires an enlarged unit cell (through broken translational symmetry or a spontaneous flux) or develops topological order. For the latter case, the momentum balance argument fixes the crystal momentum of the degenerate ground states in the different topological sectors. A useful side result of this analysis will be a general prescription to distinguish between between a Z_2 fractionalized insulator (or spin liquid) and a more conventional translation symmetry broken state, which is useful when the the order parameter for the translation symmetry breaking is not obvious. This is relevant for numerical studies on finite sized spin systems that search for fractionalized spin liquid states.[4,5]

Next, we apply the same methods to the case of exotic fermi liquids (FL^* phases) proposed recently in Ref. 6. This is phase which has conventional electron-like excitations near a fermi surface, but also possesses topological order and gapped fractionalized excitations. The question of interest here is whether the Fermi surface in these systems "violates" Luttinger's theorem (given that these phases are not continuously connected to the free electron gas this is of course not prohibited by Luttinger's proof[3]), and if so whether there is a generalization of Luttinger's theorem that can accomodate these cases as well. Indeed, applying the momentum balance argument to the FL^* phases we find that while Luttinger's theorem is violated by these Fermi volumes, this violation is not arbitrary but is constrained to be one of a few possibilities which is determined uniquely by the pattern of fractionalization.

Finally, we apply these arguments to the case of neutral superfluids. For conventional superfluids we argue that this leads to a constraint on the Berry phase acquired on adiabatically moving a vortex around a closed loop, by relating it to the boson filling. Loosely speaking this is the quantity that determined the Magnus "force" on a moving vortex. Since this relation between the Magnus force and the boson filling is obtained using the same momentum balance argument that leads to Luttinger's theorem when applied to a Fermi liquid, it may be viewed as a "Luttinger" theorem for superfluids. Alternatively, since a similar relation between boson density and Magnus force is known for superfluids with Galilean invariance,[7,8] this may be viewed as an extension of those results to the case of lattice systems. In contrast, while Galilean invariance also constrains the zero temperature value of the superfluid stiffness, that constraint does not survive the inclusion of the lattice. In order to further bring out the similarity of this relation in superluids to the Luttinger relation, we consider fractionalized superfluid phases SF^* (related to the exotic superconductor SC^* studied in Ref. 9) or equivalently superfluid analogues of the fractionalized Fermi liquid phases. We show that they too violate the conventional relation between vortex Berry phase and boson filling in exactly the same way that Luttinger's theorem is violated in FL^*. We discuss caveats in the relation between the vortex Berry phase and the boson filling in conventional and fractionalized superfluids which make the above relation less rigorous at the present time than the analogous relation for Fermi liquids and insulators.

The relation between vortex Berry phase and boson filling in lattice superconductors can lead to surprising conclusions in some cases. For example, consider a conventional superfluid (where the bosons are Cooper pairs of electrons) obtained by doping a band insulator versus another conventional superfluid obtained on doping a proximate Mott insulator. One may imagine that only the doped charges participate in the superfluidity—indeed this is roughly what is expected for a quantity like the superfluid stiffness (although it is not strictly constrained in these lattice systems[10]). However, a result of the discussion below will be that the Berry phase acquired by a vortex in this system arises from counting *all* charges in the system (and not just the charges doped into the Mott insulator). In this sense at zero temperature all particles participate in the superfluidity. In contrast, an exotic superfluid phase SF^* can display a phase where only the doped charges contribute to the Berry phase.

A recurring theme throughout this paper will be the distinction between exotic phases obtained from a correlated "band" insulator, i.e., one that has interger filling per site in the case of bosons and which could potentially form a conventional translationally invariant insulating state, versus those obtained from an exotic phase at half filling. For example, if we consider featureless Z_2 fractionalized insulators at integer and half integer filling, then at low energies they can be described by "even" and "odd" Z_2 gauge theories, respectively (in the terminology of Ref. 11). The different ground state topological sectors of the odd gauge theory can in certain geometries carry a finite crystal momentum, while the crystal momenta associated with the different ground state sectors of an even gauge theory are always zero. This distinction persists if these phases are doped to obtain exotic Fermi liquids and superfluids. The distinction is especially striking in the case of FL^*_{odd} where a Fermi volume that violates Luttinger's theorem arises. In contrast, FL^*_{even} obeys Luttinger's theorem but, is nevertheless, an exotic phase. A similar distinction will apply to the exotic superfluids—in that case the relation between the Magnus force on a vortex and the filling is the regular one for SF^*_{even} but is unconventional in the case of SF^*_{odd}.

An essential ingredient in the following arguments will be the evolution of a quantum state under flux insertion. While this recalls the argument of Laughlin[12] for the integer quantum Hall effect, there is an important distinction that must be noted. In the case of Laughlin's argument and a similar argument for the forces on superfluid vortices given by Wexler,[13] the conclusions are derived by keeping track of the change in energy during the process of flux threading. More recently, rigorous energy counting arguments for charge and spin insulators have been made by Oshikawa[14] and Hastings.[15] In contrast here we will follow Ref. 2 and rather keep track of the change in *crystal momentum* during the flux threading process, which will allow us to derive a different set of rather general conclusions that apply to a variety of phases. We also note that while the subject of Magnus force on a vortex at finite temperatures, in the presence of quasiparticle or superfluid phonon excitations, has been the subject of much lively debate (see, for example Ref. 16), our arguments will only apply to the case of zero temperature and hence cannot address any of the issues under debate.

The layout of this paper is as follows. Due to the length of the paper we give in Sec. II a simplified overview of all results and a brief indication of the method used. Then, we pass to the technical details and in Sec. III review the momentum counting procedure which will be applied to all the phases. Then, we consider conventional insulators in Sec. IV and Z_2 fractionalized insulators in Sec. V using the momentum balance argument and discuss how they may be unambiguously distinguished in numerical experiments in Sec. VI. We then review the momentum balance argument[2] for conventional Fermi liquids in Sec. VII, which leads to Luttinger's theorem, and see how this is modified in a systematic way when applied to Z_2 fractionalized Fermi liquids in Sec. VIII. Next we apply these arguments to conventional, neutral superfluids and Z_2 fractionalized superfluids, in Secs. IX and X, respectively. We conclude with some observable consequences that arise directly from these considerations.

II. OVERVIEW OF THE MOMENTUM BALANCE ARGUMENT

In this section we summarize the results of the momentum balance argument applied to different phases. While the de-

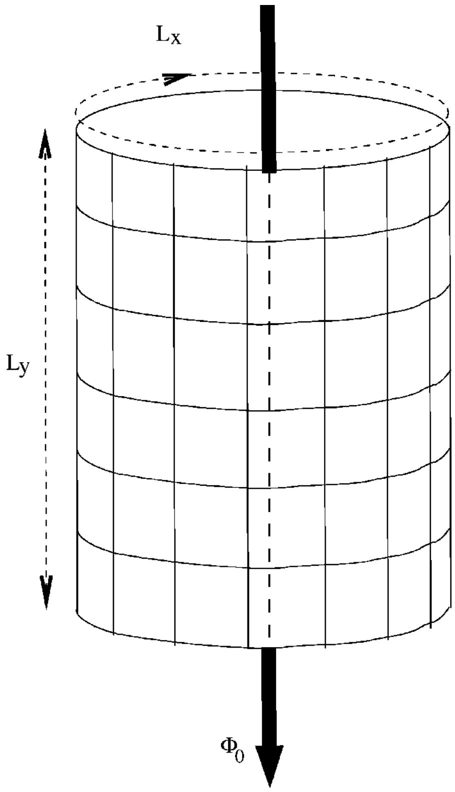

FIG. 1. Schematic figure showing flux threading in a cylinder geometry, with flux $\Phi_0 = hc/Q$.

tailed arguments leading to these results are contained in the following sections, the results themselves are easily stated, which is done below along with some heuristic supporting arguments.

Consider the system in a cylindrical geometry, as shown in Fig. 1 with dimensions L_x, L_y (integers), and a total of N particles (bosons or spinless fermions). Now imagine adiabatically threading a flux of 2π through the center of the cylinder—the particles are assumed to couple minimally to this flux with a unit charge. The total momentum imparted to the system can be calculated using Faraday's law $F_x = -1/L_x d\phi/dt$ and integrating this force over time leads to the change in momentum

$$\Delta P_x = \frac{2\pi}{L_x} N, \tag{1}$$

which is only defined modulo 2π since the system is on a lattice. A more rigorous calculation in the following sections arrives at the same result. A similar procedure performed along the perpendicular direction (for which it is convenient to think of the system living on a torus) yields the other component of the crystal momentum

$$\Delta P_y = \frac{2\pi}{L_y} N. \tag{2}$$

This is the result of trivial momentum counting—we now consider how this additional crystal momentum is accomodated in the various different phases. For later purposes, we will define the "filling" $\nu \equiv N/L_x L_y$, which is the number of particles per unit cell.

Insulators: Consider the evolution of the ground state of a many body system under the adiabatic insertion of a 2π flux. For an insulator, it may be easily verified that the final state must have an energy equal to that of the ground state, in the thermodynamic limit. That is, the system ends up either in the original state, or in a degenerate ground state. This is just because the change in energy under flux threading is related to the average current $\langle J \rangle = \frac{dE}{d\phi}$. Since in the insulating state the current must vanish in the thermodynamic limit, the energy of the final state must equal that of the initial state.

If the insulator is at integer filling, i.e., $N = \nu L_x L_y$, with the filling ν such that $\nu \in$ integer, then we have ($\Delta P_x = 2\pi\nu L_y$, $\Delta P_y = 2\pi\nu L_x$), and so $\Delta P_x \equiv \Delta P_y \equiv 0$ (mod 2π). This is compatible with the system having a unique ground state which it returns to at the end of the 2π flux threading. This is the conventional featureless insulating state (band insulator for Fermions, integer filling Mott insulator for bosons)— although in principle more complicated states are possible at integer filling as well.

The case of insulators at noninteger filling is more interesting. For definiteness, consider bosons at half filling ($\nu = 1/2$). In order that the total number of particles be an integer, we need the total number of sites $L_x \times L_y$ to be even. If we first consider the case L_y odd and L_x even, under flux insertion in the geometry of Fig. 1 we will have $\Delta P_x = \pi (\text{mod} 2\pi)$. Therefore, the initial and final states (which we have argued to be denenerate in the thermodynamic limit) must differ in crystal momentum and hence one is forced to conclude that the ground state is at least doubly degenerate in this even$\times$odd geometry. Such a degeneracy can result from one of two different reasons (we assume that time reversal symmetry is not spontaneously broken and the special case of π flux is discussed in Sec. IV D). First, the system may be heading towards translation symmetry breaking in the thermodynamic limit. In this case we can form the symmetric and antisymmetric combinations of the two ground states which clearly transform into each other under a unit horizontal translation. Translation symmetry breaking implies that there is a *local* operator that can distinguish between these two states. For example, if the system is heading towards a charge density wave state (e.g., with a stripe pattern with the charge on alternating columns in Fig. 1), then the relevant local operator is simply the charge density, which would distinguish these two states as being translated versions of one another. It may of course happen that the relevant local operator is less obvious (e.g., bond centered charge density) but nevertheless in principle this distinction between the two states can be made with some local operator. It may, however, happen that there exists no local operator that can distinguish these two states. Then, the system will appear perfectly translation symmetric, although it is an insulator at half filling. Indeed this is precisely the property of the RVB spin liquid state proposed for spin 1/2 lattice systems with one spin per unit cell—which can also be cast in the language of the above discussion on identifying the spins with hard core bosons. Therefore, these degenerate ground states can only be distinguished via a nonlocal operator. This is called topological order, where degeneracies arise on spaces with nontrivial topology that are related to creating a topological excitation which is highly nonlocal in the original variables. In the following sections we consider one concrete theoretical realization of this scenario—the case of a deconfined Z_2 gauge theory coupled to bosons carrying one half the elementary unit of charge. The translationally symmetric state at half filling may be roughly visualized as a uniform state with a half charge at each lattice site. The degenerate ground states correspond to the topological degeneracy of the theory on a cylinder—which is related to the presence or absence of an Ising flux (vison) in the hole of a cylinder. In the following sections we explicitly demonstrate that threading a 2π flux induces such a topological excitation and causes the ground state to evolve into this distinct topological sector. Thus, the way these topologically ordered states accomodate the crystal momentum imparted to the system on flux threading, despite being translationally symmetric, is by creating a vison excitation in the hole of the cylinder which then carries the appropriate crystal momentum. Later we will draw a distinction between Ising gauge theories where the vison carries a crystal momentum (odd gauge theories) and those where it does not carry momentum (even gauge theories).

Fermi liquids: The original application of the momentum balance argument was to the case of Fermi liquids in Ref. 2. This argument is reviewed in the following sections—here we just note the main points. If we begin with spinless electrons at a filling ν, then the flux threading excites quasiparticles around the Fermi surface. The total crystal momentum carried by these excitations can be converted into an integral over the volume enclosed by the Fermi surface, which leads to the relation between the filling (which enters the trivial momentum counting) and the Fermi volume. For an appropriately chosen system size, these lead to the relation

$$\nu = \frac{V_{FS}}{(2\pi)^2} + p, \tag{3}$$

where V_{FS} is the Fermi volume, and p is an arbitrary integer which represents the filled bands. This is just Luttinger's theorem—in Ref. 2 it was also applied to Kondo lattice models where it yields the large Fermi surface expected in the Kondo screened phase.

One can now ask the reverse question—given a phase whose low energy excitations are electron like Landau quasiparticles, does this phase necessarily also satisfy Luttinger's theorem? From the above momentum balance argument it is clear that in order to violate the Luttinger relation there must exist an alternate sink for the momentum. From our previous discussion of topologically ordered states, it is clear that if topological order coexists with Fermi liquid-like excitations, then the momentum balance can be satisfied with a non-Luttinger Fermi volume. In fact, the specific case of an exotic Fermi liquid with Z_2 topological order (FL^* phase) was proposed in Ref. 6 in the context of the heavy fermion systems. This phase has low energy excitations identical to that of a Landau Fermi liquid of electrons, but also a gapped Ising vortex excitation. The Luttinger relation relating the filling to the volume enclosed by the Fermi surface of these quasiparticles can then be violated, but in a very specific way. Flux threading creates a Ising vortex which carries crystal momentum π (in an odd×even system), while the remaining momentum is absorbed by the quasiparticle excitations. This leads to the modified Luttinger relation

$$\nu = \frac{1}{2} + \frac{V_{FS}}{(2\pi)^2} + p, \tag{4}$$

where p is an arbitrary integer representing filled bands. Note the crucial difference from Luttinger's relation in Eq. (3), that arises from the extra factor of $\frac{1}{2}$. Clearly, this is related to the fact that a translationally invariant insulator is possible at half filling, where the Fermi volume can shrink to zero. Thus, the difference from the original Luttinger relation is very specific, i.e., the filling of exactly half a band, for the case of Z_2 fractionalization. We have commented earlier on the difference between odd versus even Z_2 gauge theories. Again this distinction is crucial here and moreover is not directly set by the filling as it was for the case of the translationally symmetric insulating states. The above violation of the Luttinger relation only occurs in the case of the odd gauge theory, FL^*_{odd}.

Superfluids: Finally, we consider the case of neutral superfluids. Here threading a 2π flux clearly inserts a vortex through the hole of the cylinder. This can also be visualized as creating a vortex in the superfluid at the bottom of the cylinder and dragging it all the way to the top. Clearly such a vortex will experience a "Magnus force" in the direction perpendicular to its motion. Let us ignore for a moment the lattice and calculate the momentum imparted by this force $F_M = 2\pi\alpha_M \mathbf{v}\times\hat{z}$, where $\mathbf{v}$ is the velocity and α_M a constant that fixes the Magnus force. The total momentum transferred to the system is then independent of the details of the vortex motion and depends only on its net displacement—this yields $\Delta P_x = 2\pi\alpha_M L_y$. Equating this to the momentum obtained from trivial momentum counting (1) and reintroducing the lattice heuristically by allowing the the momentum to change in arbitrary integer multiples of 2π we have

$$\nu = \alpha_M + p, \tag{5}$$

where p is an arbitrary integer. Thus, the fractional part of α_M is completely determined by the boson filling ν. This can be viewed as the analogue of Luttinger's theorem for bosonic system, since it is obtained using the same line of argument. It can also be viewed as an extension of the well known equivalent result for Galilean invariant superfluids[7] (where the Magnus coefficient is the boson density) to the case of lattice superfluids. While we have been interpreting α_M above in terms of the Magnus force clearly this is not a well defined concept in a lattice system. In fact, the property that is sharply fixed by α_M is the Berry phase acquired by a vortex on adiabatically taking it around a big loop of size $\mathcal{N}$ plaquettes, which will be shown to be $2\pi\alpha_M\mathcal{N}$. This is related to the well known relation[7] in Galilean superfluids between the Magnus force and the Berry phase acquired by a vortex.

Again, one can ask if the relation in Eq. (5) can be violated in any kind of superfluid. Indeed, topologically ordered superfluid states SF^* can be defined in complete analogy with FL^*. For the particular case of an exotic superfluid state SF^* with Z_2 topological order,[9] there exists in addition to the usual vortex excitation, an Ising vortex excitation as well. Threading 2π flux then creates both a regular vortex and an Ising vortex—the latter can carry crystal momentum π (in the phase SF^*_{odd})—in which case the remaining momentum is associated with the Magnus force on the superfluid vortex. Then, in this case as well, the Magnus coefficient α_M, associated with the vortex Berry phase, satisfies

$$\nu = \frac{1}{2} + \alpha^*_M + p, \tag{6}$$

where p is an arbitrary integer. Thus, the "Luttinger relation" for a conventional superfluid in Eq. (5) is violated, in exactly the same way that FL^*_{odd} violates the Luttinger relation for conventional Fermi liquids.

III. TRIVIAL MOMENTUM COUNTING

Consider a system of N particles, each with charge Q, living on an $L_x \times L_y$ lattice wrapped into the form of a torus with periodic boundary conditions along both directions. The main result of this section is that if one adiabatically threads flux hc/Q through one of the holes of the torus, the crystal momentum difference between the initial and final state is

$$P_f - P_i = 2\pi N/L_x (\mathrm{mod}\ 2\pi) = 2\pi\nu L_y (\mathrm{mod}\ 2\pi), \tag{7}$$

where $\nu = N/(L_x L_y)$ is the charge density in units of Q (the "filling"). This result is independent of the eventual quantum phase of the system in the thermodynamic limit, and we will refer to it as the "trivial" counting. Although this has been shown in Ref. 2, we include a derivation here for the sake of completeness, and to fix notation.

The Hamiltonian for an interacting set of particles (fermions or bosons) coupled to an external vector potential can be written down as $H_A = \hat{K}_A + V[\hat{n}]$, with the kinetic energy $\hat{K}_A$, in the presence of a vector potential A_{ij}, given by

$$\hat{K}_A = -\frac{1}{2}\sum_{ij} [t_{ij} e^{-iQA_{ij}/\hbar c} B_i^\dagger B_j + h.c.]. \tag{8}$$

$B_i^\dagger$ creates a particle carrying charge Q at site i. The interaction term $V[\hat{n}]$ depends only on the density of the particles. Both t_{ij} and $V[\hat{n}]$ are invariant under lattice translations. For simplicity of presentation, we will assume that t_{ij} only connects nearest neighbor sites on a square lattice, with unit lattice spacing.

To thread a unit flux $\Phi_0 = hc/Q$ through the hole of the torus, say along the $-\hat{y}$ axis as in Fig. 1, we can choose a uniform gauge in which $A_{i,i+\hat{x}} = -\Phi(t)/L_x$ and $A_{ij} = 0$ for other links, and adiabatically increase $\Phi(t): 0 \to hc/Q$. The state we reach on flux insertion can of course be written as $|\Psi(T)\rangle = \mathcal{U}_T |\Psi(0)\rangle$ with the unitary time evolution operator $\mathcal{U}_T = \mathcal{T}_t \exp(-i\int_0^T H_A(t) dt)$ where $\mathcal{T}_t$ is the time-ordering operator. In the final Hamiltonian, the vector potential corresponds to flux $\Phi(T) = \Phi_0$.

Clearly, the initial and final wave functions, as well as the Hamiltonian, transform under gauge transformations. Thus, since the final Hamiltonian includes a unit flux quantum, we need to fix a gauge in order to consistently define the crystal momentum of a state as the eigenvalue of the unit lattice translation operator acting on the state and to compare it for the two states. We pick a gauge such that $A_{ij} = 0$ in the initial as well as the final Hamiltonian—this Hamiltonian is denoted as H_0. In this case, for a threaded flux Φ_0, we need to make a unitary gauge transformation

$$H_A(T) \to \mathcal{U}_G H_A(T) \mathcal{U}_G^{-1} = H_0 \tag{9}$$

with the operator

$$\mathcal{U}_G = \exp\left(i \frac{2\pi}{L_x} \sum_i x_i \hat{n}_i \right). \tag{10}$$

The final wave function in this gauge is, in obvious notation, $|\Psi_f\rangle = \mathcal{U}_G \mathcal{U}_T |\Psi_i\rangle$. To compute the crystal momentum of this state, we must act on it with the unit translation operator $\hat{T}$. This defines the initial and final crystal momenta, P_i, P_f through $\hat{T}|\Psi_i\rangle = \exp(-iP_i)|\Psi_i\rangle$ and $\hat{T}|\Psi_f\rangle = \exp(-iP_f)|\Psi_f\rangle$. Translating the final state we find

$$\begin{aligned}\hat{T}\mathcal{U}_G\mathcal{U}_T|\Psi_i\rangle &= (\hat{T}\mathcal{U}_G\hat{T}^{-1})(\hat{T}\mathcal{U}_T\hat{T}^{-1})\hat{T}|\Psi_i\rangle \\ &= (\hat{T}\mathcal{U}_G\hat{T}^{-1})\mathcal{U}_T e^{-iP_i}|\Psi_i\rangle,\end{aligned} \tag{11}$$

since the operator $\mathcal{U}_T$ commutes with the $\hat{T}$ as the time-dependent Hamiltonian is translationally invariant in the uniform gauge. At the same time, it is straightforward to show that

$$\hat{T}\mathcal{U}_G\hat{T}^{-1} = \exp(-i2\pi N/L_x)\mathcal{U}_G. \tag{12}$$

It is then clear that $P_f = P_i + 2\pi N/L_x (\mathrm{mod}\ 2\pi)$, or defining the filling $\nu = N/(L_x L_y)$, the change in crystal momentum is $P_f - P_i = \Delta P = 2\pi\nu L_y (\mathrm{mod}\ 2\pi)$.

It is essential for this argument to go through that one has a conserved $U(1)$ charge, this permits us to couple the charge to an inserted solenoidal flux. One can easily generalize the argument to cases where the charged particles carry spin and are coupled to spins fixed to the lattice such as in a Kondo lattice model.[2] In this case, one can thread a flux which couples to a single component of the spin of the charged carriers, and eliminate the vector potential using a unitary transformation which acts on the charged particles as well as the fixed spins.

As mentioned earlier, the result above has been derived without any assumption about the thermodynamic phase of the system. Such an assumption is important for counting the momentum in a second independent way, which provides constraints on the various quantum phases of the system and we turn to this in the remaining sections. For convenience of notation, we will set $\hbar = c = 1$ in most places.

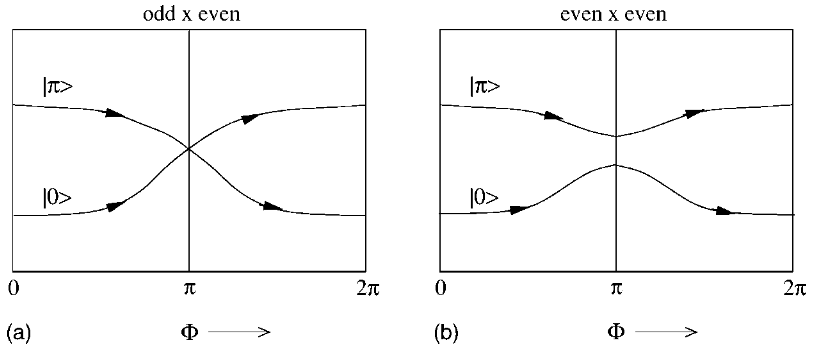

FIG. 2. Evolution of energy levels upon flux threading in a conventional insulator with twofold broken translational symmetry on a cylinder. The two levels which become degenerate ground states in the thermodynamic limit carry momenta 0, π. (a) The two levels cross upon threading flux along $\hat{y}$ in a geometry with L_x=even, L_y=odd. (b) The two levels return to themselves upon threading flux along $\hat{y}$ in a geometry with L_x=even, L_y=even.

IV. CONVENTIONAL INSULATORS

A. No broken symmetry

Consider a conventional insulator with a unique ground state and a nonzero gap to current carrying excitations. Under adiabatic flux threading, since the Hamiltonian is time dependent, the rate of change of energy is given by $\langle d\hat{H}/dt\rangle = -\Sigma_{ij}\langle \hat{J}_{ij}(t)\rangle \partial A_{ij}(t)/\partial t$ where the current operator $\hat{J}_{ij} = -iQt_{ij}(B_i^\dagger B_j e^{-iA_{ij}(t)} - h.c.)$ for the Hamiltonian with kinetic energy as in Eq. (8). Let us assume a linear rate of change of $A_{ij}(t)$ (for $j=i+\hat{x}$) over a time interval T for threading one flux quantum $\Phi_0 = 2\pi$, i.e., the electric field $E_{ij} = -\partial A_{ij}/\partial t = (2\pi/QL_xT)\hat{x}$ is a constant over the interval T. The total change in energy is thus

$$\delta E = \int_0^T dt\langle d\hat{H}/dt\rangle = 2\pi\hbar\bar{I}, \tag{13}$$

where the average current in units of Q is

$$\bar{I} = 1/(\hbar Q L_x T)\sum_i \int_0^T dt\langle \hat{J}_{i,i+\hat{x}}(t)\rangle. \tag{14}$$

Clearly $\bar{I}=0$ in an insulator in the thermodynamic limit[17]—there is no current flow, and thus $\delta E=0$!

However, if we thread one flux quantum into the system it can be eliminated using a gauge transformation which leaves the spectrum invariant, as is well known and was shown in the previous section. Since the system has a unique ground state with a charge gap, and $\delta E=0$, this means the final state and the initial state in the $A_{ij}=0$ gauge must be the same. Clearly, there is no change in crystal momentum on threading flux Φ_0, which implies

$$2\pi\nu L_y = 0 (\mathrm{mod}\, 2\pi) \tag{15}$$

for any L_y. This is only possibly if ν is an integer. Thus we arrive at the result: a *conventional* insulator with a unique ground state (i.e., no broken symmetry) and a nonzero gap to charged excitations is only possible at integer filling.[18,19]

B. Conventional insulator with broken translational symmetry

Imagine tuning the interaction $V[\hat{n}]$ in the above Hamiltonian in Eq. (8), such that the ground state of the system in the thermodynamic limit is an insulator which breaks translational symmetry. The thermodynamic ground state is clearly degenerate, the degeneracy reflecting the different broken symmetry patterns. On a finite lattice, such a system must thus have eigenstates with different crystal momentum, which, in the thermodynamic limit, become degenerate and allow us to construct linear superposition eigenstates which break the translational symmetry.

Let us consider such a system on a finite lattice, with aspect ratio such that the thermodynamic broken symmetry pattern is not frustrated. If the insulator is stabilized at a density $\nu=p/q$ (with p, q having no common factors), the flux threading argument implies the ground state must evolve under hc/Q flux insertion into a different state which has a relative crystal momentum $\Delta P = 2\pi(p/q)L_y$, with an energy equal to the ground state energy in the thermodynamic limit. These states would be "quasi-degenerate" on a large finite lattice.[20]

C. Flux threading in the conventional broken symmetry insulator

The manner in which the set of quasi-degenerate states in a broken symmetry insulator evolves under adiabatic flux insertion is fixed by momentum balance. Let us again work with a system with a twofold broken symmetry in the thermodynamic limit.

If the filling ν and L_y are such that $2\pi\nu L_y = \pi(\mathrm{mod}\, 2\pi)$, flux insertion causes a momentum change of $\Delta P_x = \pi$. This implies we must have two quasi-degenerate states differ in $\hat{x}$-crystal momentum by π, and flux threading must lead to an interchange of these two states. This is depicted schematically in Fig. 2(a) where the two states on a finite size system, denoted by $|0\rangle$, $|\pi\rangle$, begin with some splitting (which must vanish in the thermodynamic limit) and then evolve as the inserted flux Φ changes. They are degenerate and cross at $\Phi=\pi$ since the Hamiltonian is invariant under time reversal, but they cannot mix since the $\hat{x}$-crystal momenta of the two states differ by π even at this point. If the geometry is chosen such that $2\pi\nu L_y = 0(\mathrm{mod}\, 2\pi)$ these two states will no longer exchange places on threading $\Phi=2\pi$ [see Fig. 2(b)]. They no longer cross at $\Phi=\pi$ though they still carry relative π momentum.[21]

D. Local operators to detect broken symmetry states

In the presence of spontaneous translational symmetry breaking, there are local operators which can distinguish the different insulating ground states obtained in the thermodynamic limit by taking linear combinations of the degenerate momentum eigenstates as $|1\rangle=(|0\rangle+|\pi\rangle)/\sqrt{2}$ and $|2\rangle=(|0\rangle-|\pi\rangle)/\sqrt{2}$. For example, broken translational symmetry along say the $\hat{x}$ direction means it can be detected by some local Hermitian operator $\hat{O}_i$, since $\langle 1|(\hat{O}_i-\hat{O}_{i+\hat{x}})|1\rangle\neq 0$, and similarly for state $|2\rangle$. How does this manifest itself on a finite size system where such linear combinations $|1\rangle$, $|2\rangle$ are not eigenstates of the Hamiltonian?

To answer this, consider the matrix element $\langle 0|\hat{O}_i|\pi\rangle$ of the local operator between the eigenstates on the finite system. Since $\hat{O}_i$ is defined locally, it is not translationally invariant and such matrix elements will be nonzero in general on a finite system. However, knowing that $\hat{T}|0\rangle=|0\rangle$ and $\hat{T}|\pi\rangle=-|\pi\rangle$ we can rewrite this matrix element in the thermodynamic limit as

$$2(\langle 0|\hat{O}_i|\pi\rangle)=\langle 1|\hat{O}_i|1\rangle-\langle 2|\hat{O}_i|2\rangle=\langle 1|(\hat{O}_i-\hat{T}\hat{O}_i\hat{T}^{-1})|1\rangle$$
$$=\langle 1|(\hat{O}_i-\hat{O}_{i+\hat{x}})|1\rangle\neq 0. \tag{16}$$

Thus, the matrix element of this local operator $\hat{O}_i$ between states of the quasi-degenerate ground state manifold, $|0\rangle$ and $|\pi\rangle$, survives in the thermodynamic limit and implies translational symmetry breaking. The local operator could be, for instance, the energy (or charge or current) density.

This is the crucial difference between broken symmetry insulators and translationally invariant fractionalized insulators dealt with in the next section. In the latter case, matrix elements of all local operators between states forming the quasi-degenerate ground state manifold vanish in the thermodynamic limit. The system size dependence of the matrix element of local operators between states forming the quasi-degenerate ground state manifold thus distinguishes an insulator with translational symmetry breaking from a uniform fractionalized insulator. However, constructing such local operators needs some knowledge of the kind of broken symmetry, in contrast to our general conclusions in the earlier section regarding the momenta and evolution of the quasi-degenerate manifold of ground states which does not rely on such information. We return to this issue in Sec. VI.

In this section we have focused on conventional insulating states of a half filled system, and argued that they necessarily break a lattice symmetry. One case, however, needs to be looked at separately, and that is the case of exactly π flux through every elementary plaquette, which could be self-generated in the thermodynamic limit. Note, this situation can preserve time reversal symmetry, and hence should be admitted in our discussion. It is possible for such a system to be essentially an insulator at half filling, although it appears to possess translation symmetry, in that all unit cells appear identical. This issue is resolved by studying more carefully the meaning of translation invariance—it turns out that the operators that generate unit translations do not commute due to the presence of π flux in the elementary plaquette. Hence, the smallest mutually commuting translations necessarily enclose an area equal to two unit cells, and in this sense we obtain unit cell doubling. Note, the emergence of π flux per plaquette is a property that can be checked with local operators, and hence also corresponds to a conventional (nonfractionalized) state.

V. Z_2 FRACTIONALIZED INSULATOR $\mathcal{I}^*$

In the context of the insulating phase of the high temperature superconductors the question has been raised of whether a Mott insulator that breaks no symmetries could be obtained at half filling. The analogous question for our bosonic system is whether a translationally invariant insulating state can be realized at half filling. Since the hard-core boson state at half filling may be viewed as a spin $S=1/2$ system with $S_z^{\text{total}}=0$ and $U(1)$ spin rotation invariance, this is equivalent to asking whether a $S=1/2$ magnet may be in a spin liquid state. The answer, after several years of work, is *yes*, and one specific route to realizing such an insulator is via Z_2 fractionalization. The properties of such a phase[9] as well as some microscopic models which realize them are now known.[22]

The Z_2 fractionalized insulator, $\mathcal{I}^*$, is a translationally invariant insulator. It is *unconventional* in that supports gapped *fractionally charged* excitations, chargons, which carry electromagnetic charge $Q/2$ as well as a Z_2 Ising charge. These chargons interact with a Z_2 Ising gauge field in its deconfining phase. The deconfinement is reflected in the presence of yet another exotic gapped neutral excitation, the Ising vortex or "vison," which acts as a bundle of π flux as seen by the chargons which carry Ising charge.[9]

It is known that the $\mathcal{I}^*$ phase can be realized at half odd-integer or integer fillings. *Why doesn't the existence of a translationally invariant $\mathcal{I}^*$ insulator at half integer filling contradict the earlier theorem for (conventional) insulators?* The resolution of this apparent paradox is that although these insulators do not break translational invariance, the ground state of these fractionalized insulators is not unique in a multiply connected geometry (in which the flux-threading experiment is carried out). The presence of the Z_2 vortex, the vison, directly leads to a twofold degeneracy of the ground state of the system on a cylinder (fourfold on a torus). This degeneracy may be viewed as a result of having or not having a vison threading each hole of the cylinder (torus). Since the vison is a gapped excitation in the bulk of the system, there is an infinite barrier for the tunneling of the vison "string" out of the hole of the cylinder (torus) in the thermodynamic limit. Thus the vison/no-vison states do not mix in the thermodynamic limit which is crucial to obtaining leading a "topological degeneracy"—i.e., degeneracy which depends on the number of holes in the system.

At this point we introduce the following terminology for the Z_2 fractionalized insulators. The translationally symmetric Z_2 fractionalized insulator at half odd-integer filling (integer filling) will be denoted as $\mathcal{I}^*_{odd}$ ($\mathcal{I}^*_{even}$). While in the former case, translation symmetry of a half filled insulator implies that the state must be exotic, it is of course possible to have a completely conventional insulator at integer filling. Nevertheless, a Z_2 fractionalized insulator may also exist at integer filling and we refer to this as ($\mathcal{I}^*_{even}$). We will see below that these two classes of exotic insulators are in fact closely related to two classes of Z_2 gauge theories, Z_2^{odd}, Z_2^{even} in the terminology of Ref. 11.

The presence of topologically degenerate states and their evolution under flux threading allows us to satisfy the momentum balance condition. The relevant case to consider is the translationally symmetric insulator at half filling, on a cylinder with an odd number of rows. In this case, trivial momentum counting tells us that 2π flux threading leads to a degenerate state with crystal momentum π. We will argue below, this momentum is accounted for in $\mathcal{I}^*_{odd}$ since flux threading effectively adds a vison into the hole of the cylinder, which carries crystal momentum π.

A. Effective Hamiltonian for $\mathcal{I}^*$

The effective description of $\mathcal{I}^*$ is via a set of gapped charge-$Q/2$ bosons (chargons) also carrying an Ising charge, interacting with each other and minimally coupled to an Ising gauge field in its deconfining phase. In order to place the following discussion on a more concrete footing we consider a definite Hamiltonian that can describe such a system, and use it to derive properties of the states. Since we will be interested in universal properties that characte rize the state, the results themselves are more general than the particular effective Hamiltonian used. The simplest Hamiltonian which can describe a Z_2 fractionalized insulator is

$$H_A(\mathcal{I}^*) = H_g + H_m, \tag{17}$$

where

$$H_g = -K\sum_{\Box}\prod_{\Box}\sigma^z_{ij} - h\sum_{\langle ij\rangle}\sigma^x_{ij}, \tag{18}$$

$$H_m = -t_b\sum_{\langle ij\rangle}\sigma^z_{ij}(b_i^\dagger b_j e^{-iQA_{ij}/2} + h.c.) + U\sum_i (n_i - 2\bar{N})^2, \tag{19}$$

where $\sigma^{x,z}$ are Pauli matrices describing the Ising gauge fields, and $\Box$ denotes the elementary plaquette on a square lattice. The chargons, created by $b_i^\dagger$, are minimally coupled to the Ising gauge field, as well as to the external vector potential A_{ij} with electromagnetic charge $Q/2$. The second term in H_m describes repulsion between chargons at the same site.

The Hamiltonian (19) has a local Z_2 invariance under the transformation $b_i \to \alpha_i b_i$ and $\sigma^z_{ij} \to \alpha_i \sigma^z_{ij}\alpha_j$ where $\alpha_i = \pm 1$. Such gauge rotations are generated by unitary transformations using the operator $\hat{G} = \Pi_i \hat{G}_i$ with

$$\hat{G}_i = \exp\left[i\frac{\pi}{4}(1-\alpha_i)\left(\sum_{j=nn(i)}\sigma^x_{ij} + 2n_i\right)\right]. \tag{20}$$

Local Z_2 invariance implies that $[\hat{G}_i, H_A(\mathcal{I}^*)] = 0$. Since we wish to work with eigenstates of $H_A(\mathcal{I}^*)$ which are invariant under such gauge transformations, translationally invariant physical states have to satisfy

$$\hat{G}_i|\text{phys}\rangle = (\pm 1)|\text{phys}\rangle. \tag{21}$$

Let us choose $\hat{G}_i = +1$ everywhere.

It is instructive to first consider the limit h, $U \gg K$, t_b. In this case, since $h \gg K$, the gauge theory is confining. Depending on the filling, it is then possible to show that one recovers conventional insulating phases such as a uniform band insulator (for $\bar{N}$ = even integer), or broken symmetry states such as bond-centered (with $\bar{N}$ = odd integer, and $U \gg h$) or site-centered (with $\bar{N}$ = odd integer and $h \gg U$) charge density wave states. Thus, the above effective Hamiltonian in this limit is capable of describing well understood conventional insulators.

However, this Hamiltonian has a richer phase diagram. The parameter regime where an exotic fractionalized insulator is expected for the above Hamiltonian is easily determined. For $K \gg h$, the Ising gauge field will be in its deconfining phase, so we can pick[23] $\sigma^z_{ij} \approx 1$. Similarly, since we are interested in the insulating phase, let us work in the limit of large chargon repulsion $U/t \gg 1$ with $2\bar{N}$, which is twice the filling fraction of the charge Q bosons, being an integer. In this limit, it is clear that it is energetically favorable to also set the chargon number $n_i = 2\bar{N}$ at each site (which is possible since $2\bar{N}$ is an integer) as a starting point to understand the insulator. The density of charge-Q bosons in the insulator is just $\nu = \bar{N}$, and ν could thus either be an integer or a half odd integer in the $\mathcal{I}^*$ phase corresponding to even/odd integer values of $2\bar{N}$.

In the above regime of parameters, the system clearly has a charge gap $\mathcal{O}(U)$ for adding an n_i particle (chargon) which is a charge-$Q/2$ and Ising-charged excitation that can propagate freely (since the gauge field is deconfined). It also has an energy gap $\mathcal{O}(K)$ to changing $\sigma^z_{ij} \to -1$ on a bond which changes $\Pi_{\Box}\sigma^z_{ij} \to (-1)$ on adjacent plaquettes corresponding to creating gapped visons. It thus describes an exotic insulator.

B. Flux threading in $\mathcal{I}^*$

Below, we will consider the effect of threading 2π flux on the Z_2 fractionalized insulators in the cylindrical geometry, using the effective Hamiltonian (19). This will be done in two steps. We first consider the limit of being deep in the fractionalized phase [i.e., set the vison hopping to zero; $h=0$ in Eq. (19)] where it can be easily argued that 2π flux threading leads to the insertion of a vison through the hole of the cylinder. The momentum balance argument then allows us to read off the crytal momenta of the visons in the different situations. Then, we turn back on a finite vison hopping $h \neq 0$, and use continuity arguments to conclude that these crystal momenta assignments remain unchanged.

1. Flux threading with static visons

Consider at first the limit of being deep in the fractionalized phase $h/K \to 0$ by setting $h=0$ identically (i.e., no vison hopping), so that we can choose $\sigma^z_{ij} = 1$ everywhere.[23] Let us adiabatically thread flux 2π in the $\hat{y}$ direction for the above system on a cylinder such that the starting from the initial eigenstate $|\Psi(0)\rangle$ in the absence of flux, the final state

$$|\Psi(T)\rangle = \mathcal{U}_T|\Psi(0)\rangle, \tag{22}$$

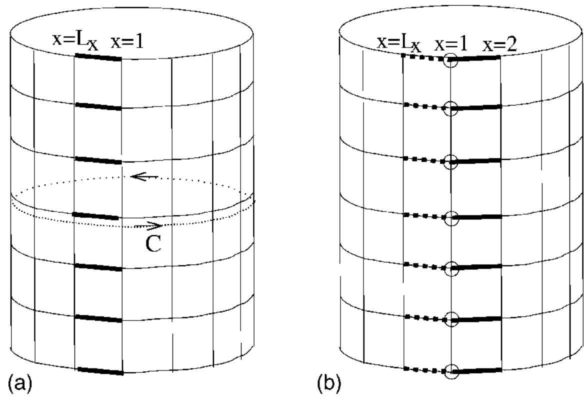

FIG. 3. (a) Schematic figure showing a vison threading the hole of a cylinder in the absence of vison tunneling terms. The dark (light) bonds correspond to $\sigma^z_{ij}=-1(\sigma^z_{ij}=+1)$. We can detect the presence of the vison by evaluating the Wilson loop operator $\Pi_C\sigma^z_{ij}$ along the contour C taken around the cylinder. (b) The translation operator along the $\hat{x}$ direction moves the dark ($\sigma^z_{ij}=-1$) bonds by one lattice spacing from (L_x, 1) to (1, 2) at each y. This is accomplished equivalently by acting with the gauge transformation operator G_i (which changes the sign of σ^z_{ij} on all bonds emanating from i) acting on each of the circled sites.

$$\mathcal{U}_T = \mathcal{T}_t \exp\left(-i\int_0^T H_A(\mathcal{I}^*,t)dt\right), \tag{23}$$

where $\mathcal{T}_t$ is the time-ordering operator. We can go to the $A_{ij}=0$ gauge by making a unitary transformation $H_A(\mathcal{I}^*,T) \to \mathcal{U}_G H_A(\mathcal{I}^*,T)\mathcal{U}_G^{-1} \equiv H_0(\mathcal{I}^*)$ (corresponding to zero flux).

Since the chargons carry a charge $Q/2$, the hc/Q flux quantum threading the cylinder appears as an Aharonov-Bohm flux of π for the chargons. The gauge transformation which returns the Hamiltonian to its original form thus also acts on the Ising gauge fields to remove this extra π flux. Hence we have

$$\mathcal{U}_G = \mathcal{U}_\phi \mathcal{U}_\sigma, \quad \text{with} \tag{24}$$

$$\mathcal{U}_\phi = \exp\left(i\frac{\pi}{L_x}\sum_i x_i \hat{n}_i\right), \tag{25}$$

$$\mathcal{U}_\sigma = \prod_{ij\in\text{cut}} \sigma^x_{ij} \tag{26}$$

and "cut" refers to the set of links for which $x_i=L_x$, $x_j=1$ (shown in Fig. 3). Thus, the final state in the $A_{ij}=0$ gauge is $|\Psi_f\rangle = \mathcal{U}_\phi\mathcal{U}_\sigma\mathcal{U}_T|\Psi_i\rangle$. Since the system is an insulator, the final state on threading flux Φ_0 must be one of the states which forms part of the degenerate ground state manifold in the thermodynamic limit.

Let us define the loop operator $W_C=\Pi_C\sigma^z_{ij}$ where the loop C is taken around the cylinder (see Fig. 3). Clearly, since $h=0$, this operator commutes with the Hamiltonian (19). We can use this operator to check whether there is a vison through the hole of the cylinder. Namely, if we are in a (reference) state with $\sigma^z_{ij}=1$ everywhere, then $W_C=1$ and this is the no-vison state $|v=0\rangle$. If on the other hand $W_C=-1$ for each loop C around the cylinder, we conclude that there must be a vison threading the hole, and we shall refer to this as $|v=1\rangle$. Let us evaluate W_C for the two states $|\Psi_{i,f}\rangle$ above. We easily find $W^i_C=1$ in the initial state. In order to find the eigenvalue of W_C in the final state, we note that since $h=0$, the initial assignment of σ^z_{ij} does not time evolve, and we only need to evaluate the effect of the unitary transformation $\mathcal{U}_G$ on W_C. This yields the result that the eigenvalue in the final state is: $W^f_C=-1$.

Thus, for $h=0$, threading a 2π flux adds a vison to the hole of the cylinder and interchanges the two ground states on the cylinder, $|v=0\rangle \leftrightarrow |v=1\rangle$. (In fact, in the absence of dynamical matter fields, i.e., $t_b=0$, the operator $U_\sigma \equiv \Pi_{ij\in\text{cut}}\sigma^x_{ij}$ commutes with the Hamiltonian and can be viewed as the "vison creation operator" introducing a vison into the hole of the cylinder and changing the sign of W_C.)

Momentum balance then tells us that adding a vison into the hole of the cylinder must change the momentum of the system by $2\pi\nu L_y$. The only situation where this is a nontrivial crystal momentum is for the case of $\mathcal{I}^*_{odd}$ on a cylinder with an odd number of rows (L_y odd). Then we expect the two states $|v=0\rangle$ and $|v=1\rangle$ to differ by crystal momentum π. In all other cases, i.e., for L_y even, or of $\mathcal{I}^*_{even}$, the vison carries no momentum.

As we shall see in the next subsection, this is consistent with a direct computation of the vison momentum in the pure Ising gauge theory. We now switch back on the vison hopping $h\neq 0$ and ask how these conclusions might be affected.

2. Vison state in the presence of dynamical gauge fields

Turning on a nonzero h, gives dynamics to the gauge field. In this case the loop product W_C no longer commutes with the Hamiltonian; we cannot use its eigenvalues to label the states. Let us first see what effect this has on $\mathcal{I}^*_{even}$. The two states $|v=0\rangle$ and $|v=1\rangle$ both carry zero crystal momentum, and will now mix to give eigenstates of the Hamiltonian. Thus, on flux threading there is no level crossing—threading a Φ_0 flux returns us to the original ground state. For $\mathcal{I}^*_{odd}$ on a cylinder with even L_y, the two low lying states carry zero crystal momentum, and a similar conclusion applies.

The situation is more interesting for $\mathcal{I}^*_{odd}$ on a cylinder with an *odd* L_y. Now, the states $|v=0\rangle$ and $|v=1\rangle$ cannot mix since they carry different momenta. Thus, even in the presence of $h\neq 0$ (so long as we remain in the same phase), we can continue to distinguish them and we can continue to label them as no-vison/vison states by their momentum, although they are not eigenstates of the W_C operator any longer. In this case, the crossing of the two states on threading a Φ_0 flux continues to occur, since the crystal momentum must change by π in order to satisfy momentum balance. Thus we may conclude that for the case of $\mathcal{I}^*_{odd}$ on an odd length cylinder, the two degenerate ground states (no-vison and vison through the hole of the cylinder) differ by crystal momentum π. This is the result of the momentum balance argument applied to Z_2 fractionalized insulators. We will sometimes simply refer to this result for $\mathcal{I}^*_{odd}$ as "the vison carrying momentum π per row of the cylinder," omitting to point out each time that the vison in question lives in the hole of the cylinder.

The above result is consistent with the vison momentum computed using: (i) the pure Ising gauge theory (as shown in the next subsection), (ii) variational wave functions for Z_2 spin liquids (as shown in Sec. V D), and, (iii) arguments presented for short-range dimer models[24] for $\mathcal{I}^*_{odd}$.

A side result of this analysis of identifying the vison crystal momenta in various situations is an unambiguous way of distinguishing fractionalized states from states with translation symmetry breaking for the half filled insulator. This is described in detail in Sec. VI.

C. Vison momentum computed directly in the pure Ising gauge theory

In order to check our deduction about the vison momentum, let us directly compute this quantity in a pure Z_2 Ising gauge theory without dynamical matter fields. If the charge gap in the insulator is large, this is the effective description of the insulator $\mathcal{I}^*$. Namely, in the limit $U \gg t_b$ in the Hamiltonian (19) and for integer values of $2\bar{N}$, a good caricature of the insulating state is to set $n_i = \bar{N}$ at each site and only consider fluctuations of the Ising gauge fields. This reduces the constraint on the physical Hilbert space to

$$\hat{G}_i^{red}|\text{phys}\rangle = (-1)^{2\bar{N}} \exp\left[i\frac{\pi}{4}(1-\alpha_i) \sum_{j=nn(i)} \sigma^x_{ij} \right]|\text{phys}\rangle = |\text{phys}\rangle \tag{27}$$

or equivalently, focusing only on the nontrivial case of $\alpha_i = -1$

$$\prod_{j=nn(i)} \sigma^x_{ij} = (-1)^{2\bar{N}} \tag{28}$$

in the subspace of physical states.

Again, if we begin with $h=0$, one ground state $|v=0\rangle$ of the gauge theory on a cylinder may be obtained as the reference state $\sigma^z_i = 1$ projected into the physical subspace, and for this one has the loop operator $W_C = 1$. A second (degenerate) state may be obtained by acting on this ground state with

$$V^\dagger_{L_x,1} = \prod_{ij \in \text{cut}} \sigma^x_{ij} \tag{29}$$

which commutes with the H_g for $h=0$. The subscripts $(L_x, 1)$ on $V^\dagger$ are a mnemonic for the column on which the σ^x operators act as shown in Fig. 3(a). The resulting state has $W_C = -1$. Let us compute the momentum of these two states. Clearly, the state $|v=0\rangle$ has zero momentum since it is translationally invariant by construction. To compute the momentum of the second state, we first note [see Fig. 3(b)] that

$$\hat{T} V^\dagger_{L_x,1} \hat{T}^{-1} = V^\dagger_{1,2} \tag{30}$$

$$= \left[\prod_i \Big(\prod_{j \in nn(i)} \sigma^x_{i,i+\hat{x}} \Big) \right] V^\dagger_{L_x,1}, \tag{31}$$

where the sites i have $x_i = L_x$ and correspond to the circled sites in Fig. 3(b). Using the constraint in Eq. (28), this reduces to

$$\hat{T} V^\dagger_{L_x,1} \hat{T}^{-1} = \left(\prod_i (-1)^{2\bar{N}} \right) V^\dagger_{L_x,1} = \exp(i2\pi \bar{N} L_y) V^\dagger_{L_x,1}. \tag{32}$$

Thus, for the second state, $|v=1\rangle = V^\dagger_{L_x,1}|v=0\rangle$, acting with the translation operator leads to

$$\hat{T}|v=1\rangle = (\hat{T} V^\dagger_{L_x,1} \hat{T}^{-1}) \hat{T}|v=0\rangle, \tag{33}$$

$$= \exp(i2\pi \bar{N} L_y) V^\dagger_{1,2}|v=0\rangle, \tag{34}$$

$$= \exp(i2\pi \bar{N} L_y)|v=1\rangle. \tag{35}$$

In other words,(i) for even $2\bar{N}$, namely in $\mathcal{I}^*_{even}$, the state $|v=1\rangle$ carries zero crystal momentum and , (ii) for odd $2\bar{N}$, namely in $\mathcal{I}^*_{odd}$, the vison state $|v=1\rangle$ carries momentum πL_y.

As before, we can now turn on a nonzero field h. In $\mathcal{I}^*_{even}$, the two ground states will mix and split in a finite system, since they carry the same momentum quantum number. The same is true for $\mathcal{I}^*_{odd}$ with even L_y. However, for $\mathcal{I}^*_{odd}$ with odd L_y, the two ground states carry relative momentum π, thus they cannot mix even on a finite system with nonzero h and can be distinguished by their momentum.

Finally, we may introduce dynamical matter fields. Although the operator in Eq. (29) no longer can be identified as a vison creation operator, the low energy structure of the system, i.e., topological degeneracies, will not change as long as we are in the same phase. Also, since the crystal momenta of these low lying states on the cylinder can only be one of 0, π (from time reversal invariance), continuity requires that the crystal momentum assignments made before for the low lying states continue to hold in the presence of dynamical matter fields. This constitutes a direct check of the results deduced above using momentum balance arguments.

D. Momentum computation from variational wave functions for the vison

So far, we have discussed bosonic models for the insulators $\mathcal{I}^*$. However, as mentioned earlier, we can view a hardcore boson as a $S=1/2$ spin, and the insulating state $\mathcal{I}^*_{odd}$ with $2\bar{N}$ an odd integer as a Z_2 fractionalized spin liquid insulator. Such spin liquid insulators have long been of interest in connection with frustrated magnets and the high temperature superconductors. The $Q=1/2$ chargon excitations in the bosonic language correspond to $S=1/2$ excitations (called spinons) in the spin liquid. What do the visons in $\mathcal{I}^*_{odd}$ correspond to?

To answer this, we note, following Anderson,[25] that one can represent of the ground state wave function for spin liquids with short-range antiferromagnetic correlations by "Gutzwiller projecting" a superconducting wave function, i.e., restricting to configurations with a fixed number of electrons per site. Such a picture also emerges from mean-field studies of frustrated magnets using a fermionic representation for the spins. This suggests that perhaps excitations of the spin liquid may also be related to excitations in the superconductor. Following this line of thought, the $S=1/2$ spinon in the spin liquid may be viewed as a projected Bogoliubov quasiparticle of the superconductor. Similarly it is natural to expect that the $hc/2e$ vortex in the superconductor becomes the vison.[26]

We can check this possibility by computing the momentum of a projected $hc/2e$ Bardeen-Cooper-Schrieffer (BCS) vortex threading the cylinder, with odd/even number of electrons at each site, and seeing if it agrees with the results for $\mathcal{I}^*_{odd}/\mathcal{I}^*_{even}$ obtained above. To do this, we write the BCS state (with total electron number N_e) as

$$|BCS(N_e)\rangle = \left(\sum_{\mathbf{k}} \phi_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k},\downarrow}\right)^{N_e/2}, \tag{36}$$

where $\phi_{\mathbf{k}}$ denotes the internal pair state of the Cooper pair formed by $(\mathbf{k},\uparrow)$ and $(-\mathbf{k},\downarrow)$, which carries zero center of mass momentum.

To get the spin liquid state at half filling, we have to choose $N_e=N$, the number of lattice sites, and Gutzwiller project this state by acting with the operator $P_G=\Pi_i(1-n_{i\uparrow}n_{i\downarrow})$ which eliminates configurations in which two electrons occupy the same site. The variational ansatz for the spin liquid ground state is then $P_G|BCS(N)\rangle$, and it is a translationally invariant state with zero momentum. To construct a BCS $hc/2e$ vortex threading the cylinder, we need to pair states with $(\mathbf{k}+\mathbf{q}/2,\uparrow)$ and $(-\mathbf{k}+\mathbf{q}/2,\downarrow)$ with $\mathbf{q}=(2\pi/L_x)\hat{x}$ and amplitude $\phi_{\mathbf{k}}$; this leads to the N-particle vortex state $|hc/2e(N)\rangle$ carrying a momentum of $\mathbf{q}$ per pair, or a total momentum $(2\pi/L_x)(N/2)$. Setting $N=L_xL_y$, we see that the momentum of this state is just πL_y. The projection operator P_G commutes with the translation operator. Thus the trial vison state $|v\rangle=P_G|hc/2e(N)\rangle$ has a momentum πL_y in agreement with earlier arguments for the $\mathcal{I}^*_{odd}$ insulator.

With an even number of electrons at each site, the vison wave function carries no crystal momentum. This is consistent with our earlier result for $\mathcal{I}^*_{even}$.

VI. IDENTIFYING Z_2 FRACTIONALIZED STATES IN NUMERICAL EXPERIMENTS

Numerical investigation of microscopic models, for example, exact diagonalization studies,[4,5] are an important tool in finding new states of matter such as states with Z_2 topological order. In this context it is important that reliable diagnostics be available for the identification of these states in the system sizes that can currently be solved on the computer. This question may seem straightforward in principle, the fourfold topological degeneracy of the Z_2 states on a torus that are indistinguishable by any local operator seem to provide a unique prescription. However, in practice there are several potential problems. First, since the numerical simulations are performed on finite sized systems, states that are degenerate in the thermodynamic limit are only approximately so in these systems. The problem is particularly severe when gapless Z_2 charged matter fields are present, in which case the splitting between topological sectors that differ by the presence of a vison can be large and go down to zero only algebraically with system size (in contrast to the exponentially small splitting in the absence of such gapless gauge charged matter fields). Second, if degeneracies arise as a result of broken translation symmetry, rather than topological order, the relevant order parameter for this translation symmetry breaking may be hard to identify, and hence we would like to have available a prescription for distinguishing such states even if the order parameter is not known. Below we use insights from the momentum balance arguments to resolve both of these issues. Indeed we will see that the analysis of the previous sections, with their focus on finite sized systems and crystal momentum quantum numbers, are ideally suited to addressing these questions. We begin by addressing the second of these two questions first—i.e., given a set of states comprising the low energy manifold of the system as the thermodynamic limit is approached, how does one distinguish topologically ordered states from a conventional translation symmetry broken state?

A. Z_2 Topological order versus translation symmetry breaking

Consider a system of bosons on a lattice at half filling (or equivalently a spin 1/2 system with one spin per unit cell). As discussed previously, a translationally invariant insulating phase implies the presence of topological order (although it is possible to have topologically ordered phases that also break translation symmetry). Assume that a group of low lying states have been identified—under what conditions can we associate these with the degeneracies assocated the Z_2 topological order, rather than with low lying states leading to translation symmetry breaking?

First consider the system in the cylindrical geometry with an odd number of rows (L_y odd, L_x even in Fig. 1). Then, flux threading ensures that we will have two low lying states with crystal momentum $P_x=0$, $P_x=\pi$ which are interchanged on threading 2π flux. This is irrespective of whether the system is heading towards translation symmetry breaking or towards a Z_2 topologically ordered state in the thermodynamic limit. Therefore this setup is not particularly useful for discussing for distinguishing the two states. One may also consider the toroidal geometry for an even $\times$ odd system, however, this can potentially frustrate certain patterns of translation symmetry breaking in a half filled system and hence we do not consider it further here.

Now consider the system in a toroidal geometry, but with both L_y, L_x *even*. Now, we have shown earlier that a Z_2 topologically ordered state will have four low lying excitations, all with zero crystal momentum in this geometry. A conventional translation symmetry breaking state, on the other hand, will invariably have at least one state in the low energy manifold that carries nonzero crystal momentum, in addition to a zero crystal momentum state. This follows almost by definition, in order to build up a translation symmetry breaking state one needs to make a linear combination of states with different crystal momenta. This then is a precise way to tell apart a Z_2 topological state from a more conventional translation symmetry breaking state, which just requires a correct identification of the low energy manifold and the crystal momenta of these states. The translationally symmetric topologically ordered state is present if there are four low lying states with zero crystal momenta. If topological order coexists with translation symmetry breaking, then too this quadruplet of zero momentum states persist, although other quadrupled states with different crystal momenta will be present in the low energy manifold. This is true for both $\mathcal{I}^*_{even}$ and $\mathcal{I}^*_{odd}$.

Recent exact diagonalization studies of a multiple spin exchange model on a triangular lattice have found signatures of an interesting new spin state, which has been proposed to be a topologically ordered spin liquid phase in a certain regime of parameters.[4] Let us apply the method of distinguishing topological order from broken translational symmetry discussed above, to these states.

In the parameter regime of interest, the system in Ref. 4 was argued to be heading, in the thermodynamic limit, towards a spin gapped phase without long-range magnetic order. Furthermore, a set of three spin singlet states which appear to become degenerate with the ground state with increasing system size were identified. The authors were unable to find simple valence bond (e.g., nearest neighbor) crystal states that would lead to degenerate ground states with the quantum numbers (crystal momenta, rotations, reflections) of these low lying states. Hence they identified this apparent fourfold degeneracy with the degeneracy arising from topological order of a Z_2 fractionalized spin liquid, such as described by Eq. (19), on a torus. While this would be a very interesting result, we can ask if the quantum numbers of these nearly degenerate states are consistent with those of a vison in an odd Ising gauge theory, that we have derived earlier. However, the three excited states which appear to become degenerate with the ground state carry *nonzero* crystal momentum on a 6×6 lattice.[4] This is in disagreement with our conclusion regarding vison states in a Z_2 fractionalized phase,[27] namely, that they carry zero crystal momentum on even$\times$even lattices. We therefore conclude that this interesting identification of Z_2 topological order in these systems does not stand up to detailed scrutiny. The actual nature of the phase being approached by these systems then remains an open question, especially since an extensive search of conventional broken symmetry states in Ref. 4 did not yield a candidate phase. The remaining possibilities are perhaps a conventional translational symmetry broken valence bond crystal phase, involving non-nearest neighbor dimers, some other more exotic fractionalized state, or that the all the low lying states associated with the broken symmetry have not been identified as a consequence of finite size effects. Note, the evolution of these states under flux threading which they have studied on odd$\times$even and even$\times$even lattices is also consistent with a conventional broken symmetry state.

B. Eliminating the spinon contribution to vison splitting

In this subsection we will utilize the flux threading proceedure to find a way of eliminating the splitting between the topologically "degenerate" states that arises from the presence of Z_2 charged matter fields. While in the thermodynamic limit, a Z_2 fractionalized system, must possess a fourfold ground state degeneracy on the torus and a twofold ground state degeneracy on the cylinder, in a finite system this spiltting may be so large that it makes the identification of the low energy manifold problematic. In this subsection we will utilize the flux threading proceedure to find a way of eliminating the part of the splitting between these states that arises from the presence of Z_2 charged matter fields.

In order to study the problem in more detail consider a finite sized system in the cylindrical geometry that is heading towards a Z_2 fractionalized insulating state. Consider first the situation on an even$\times$even lattice. The low energy manifold consists of a pair of states that eventually become degenerate in the thermodynamic limit, but at this stage have a finite splitting ΔE. The spitting arises from two sources: first there is vison tunneling, that mixes the zero and one vison states, which acting alone, would lead to a splitting of ΔE_{hop}. Since this involves a gapped vison (with gap ϵ) hopping across the entire height L_y of the cylinder, one would expect this to be exponentially small in their product, i.e., $\Delta E_{\text{hop}} \propto \exp(-cL_y\epsilon)$, where c is a constant. The second contribution to the splitting arises from the presence of matter fields that carry gauge charge. Clearly, the presence or absence of a vison will affect the propagation of these particles and in the absence of vison tunneling will give rise to an energy splitting ΔE_{mat}. With gapped Z_2 gauge charged matter fields (with a gap e), clearly this splitting will require virtual processes where the gapped particle goes once around the cylinder which implies $\Delta E_{\text{mat}} \propto \exp(-c' L_x e)$. The total splitting is easily seen to be: $\Delta E = \sqrt{\Delta E^2_{\text{hop}} + \Delta E^2_{\text{mat}}}$. Thus, in situations like the one described above, where both visons and gauge charged matter have a healthy gap, finite sized system studies can in practice isolate the low energy multiplet that leads to topological degeneracy in the thermodynamic limit.[9] Note that the presence of gapless matter fields which are *gauge neutral* do not affect these conclusions. Further, at least in principle, the topological degeneracies described here, with splittings which are exponentially small in system size, can be separated from low lying modes of the gauge neutral excitations whose splitting scales inversely with system size.

However, if the *gauge charged* matter is gapless (e.g., if there are fermionic Z_2 gauge charged excitations with a Dirac spectrum that often appear in mean field theories of spin liquids) then the splitting of the zero and one vison states are no longer exponential in the perimeter size of the system, but only a power law, i.e., $\Delta E_{\text{mat}} \propto L_x^{-\eta}$, where $\eta > 0$, and this dominates ΔE. This is potentially a serious problem since in a finite sized system the splitting is very likely to be large and also hard to distinguish from the low energy states arising from the gapless fermions, which also have energies that vanish as the inverse size of the system. Below we will prove that in the presence of π flux (i.e., antiperiodic boundary conditions for the unfractionalized bosons/magnons) the matter contribution to the vison splitting is switched off! Essentially this arises because the Z_2 charged particles, which also carry half a unit of charge, see the antiperiodic boundary conditions as a flux of (say) $-\pi/2$. Adding a vison then implies a flux of $\pi/2$. However, since these two situations are related by time reversal symmetry, the energy contribution from the matter fields in these two cases is identical—which implies that the splitting arises solely due to vison tunneling, which can be made exponentially small.

In order to show that at a flux of π the matter contribution to the vison splitting vanishes $\Delta E_{\mathrm{mat}}=0$, we adopt the following procedure. We consider the system (heading towards a Z_2 fractionalized state) on an even$\times$even cylinder, and consider first the limit where the vison hopping is turned off. Then, the splitting of the low energy states occurs entirely because of the gauge charged matter contribution. We now consider introducing a vison through the hole of the cylinder and argue below that exactly at flux π the two states are exactly degenerate (even in a finite system). Since the vison hopping has been tuned to zero, the remaining source of splitting (arising from the gauge charged matter fields) must also be zero at this point. We can then reintroduce the vison hopping, and the states at flux π will now be split, but the splitting occurs entirely from vison tunneling.

Let us now show that in the absence of vison tunneling, an exact degeneracy occurs at π flux. We consider for definiteness the model in Eq. (19)—although it contains gapped matter fields with Z_2 gauge charge, the conclusions simply show that all matter field contributions cancel at this special flux, and hence can be easily extended to the case of gapless matter fields as well. We consider the limit of vanishing vison tunneling [i.e., $h=0$ in Eq. (19)]. We start with zero flux through the cylinder and consider inserting, adiabatically, a flux of 2π.

Then, as argued in Sec. V, inserting a 2π flux leads to insertion of a vison. Now, at zero flux, the two low lying states can be classified in terms of vison number, since the vison hopping has been set to zero. The vison number is measured by the operator $W_C=\Pi_C\sigma^z_{ij}$ where the loop C is taken around the cylinder (see Fig. 2). Clearly, since $h=0$, this operator commutes with the Hamiltonian, and the two low lying states can be labeled with the eigenvalues of $(1-W_C)/2$, i.e., the vison number. The splitting between these levels arises entirely from the gauge charged matter fields. On flux threading, these two states must then interchange—since threading 2π flux inserts a vison in this limit. This means that the two levels have to cross at some point (or more generally at an odd number of points) as a function of flux. Now, time reversal symmetry tells us that if there is a crossing point at flux ϕ, then there must also be one at the point $2\pi-\phi$. Thus, in order to arrange for an odd number of crossing to ensure the levels do interchange, we need that there is always a crossing at flux π. Thus, the two states with vison and no vison are exactly degenerate at this value of the flux and hence we conclude that the splitting from the gauge charged matter fields vanishes at this value of flux. Now, turning on the vison hopping $h\neq 0$ will lead to a finite splitting even at π flux, but this splitting arises entirely from vison tunneling and hence at this value of the flux we have $\Delta E=\Delta E_{\mathrm{hop}}$ and hence vanishes exponentially in the width of the system. These arguments can easily be taken over to the toroidal geometry as well.

We note that this result is useful even in the study of SU(2) symmetric spin liquid states,[4] where although intoducing the π flux will require breaking the SU(2) symmetry, this only occurs along one row of the cylinder (e.g., changing the sign of the exchange constants for the S_xS_x and S_yS_y interactions), and hence may be viewed as a fairly weak perturbation away from full SU(2). It should also be useful in projected wave function studies, especially in establishing the existence of Z_2 fractionalized states with excitations that have a Dirac dispersion.[26]

Finally we note that while turning on a flux of π is effective in canceling the splitting arising from dynamical matter fields, the splitting from vison tunneling can be canceled in a like manner by considering a cylinder with an odd number of rows at half filling, where the vison and no vison states differ by crystal momentum π and hence do not mix. When both these processes are active we expect the vison and no-vison states to be exactly degenerate. Indeed, this is borne out by the observation that for a cylinder with an odd number of rows at half filling, when the threaded flux reaches π there is always a level crossing just from momentum balance arguments and time reversal symmetry [see Fig. 2(a)].

VII. MOMENTUM BALANCE FOR FERMI LIQUIDS

A. Conventional Fermi liquids

Let us first briefly review the momentum balance argument due to Oshikawa[2] for conventional Fermi liquids where it leads to Luttinger's theorem. Consider fermions with charge Q and spin $\uparrow,\downarrow$ at a filling per site of $\nu_\uparrow=\nu_\downarrow=\nu$. Now consider flux threading in the cylindrical geometry of Fig. 1 with L_x columns and L_y rows. We imagine threading unit flux $\Phi_0=hc/Q$ that only couples to the $\uparrow$ spin fermions. Via trivial momentum counting this proceedure can be seen to impart a crystal momentum of

$$\Delta P_x = 2\pi\nu_\uparrow L_y. \tag{37}$$

Similarly, one could imagine performing the flux threading with the cylinder wrapped along the perpendicular direction which would yield a crystal momentum change

$$\Delta P_y = 2\pi\nu_\uparrow L_x. \tag{38}$$

Now in the regular Fermi liquid phase, this crystal momentum imparted during flux threading is accounted for entirely by quasiparticle excitations that are generated near the Fermi surface. Using the fact that long lived quasiparticles exist near the Fermi surface, and the fact that the Fermi liquid is adiabatically connected to the free Fermi gas, the quasi-particle population δn_p excited during the flux threading procedure can be worked out. Clearly, flux threading for noninteracting fermions will lead to a uniform shift of the Fermi sea by $\Delta p_x=2\pi/L_x$ from which the quasiparticle distribution function can be determined. Indeed all of these excitations are close to the Fermi surface, which is required in order to apply Fermi liquid theory. The total crystal momentum carried by these excitations can be written as

$$\Delta P = \sum_p \delta n_p \mathbf{p}. \tag{39}$$

It is convenient to first evaluate this expression neglecting the discrete nature of allowed momentum states in a finite volume system and treating the shift in the Fermi sea $\delta p_x = 2\pi/L_x$ as infinitesimal. This yields

$$\Delta P_x = \oint_{FS} p_x \frac{\delta \mathbf{p} \cdot d\mathbf{S}_p}{\frac{2\pi}{L_x}\frac{2\pi}{L_y}}, \tag{40}$$

where $d\mathbf{S}_p$ is a vector normal to the Fermi surface, and the integral is taken around the Fermi surface. Using Gauss divergence theorem this can be converted into an integral over the Fermi volume which yields

$$\Delta P_x = \delta p_x \int_{FV} \frac{dV}{\frac{2\pi}{L_x}\frac{2\pi}{L_y}} \tag{41}$$

thus,

$$\Delta P_x = \frac{2\pi}{L_x} \frac{V_{FS}^{\uparrow}}{\frac{2\pi}{L_x}\frac{2\pi}{L_y}}.$$

A more careful derivation that keeps track of the discreteness of the allowed momenta gives the same result. Clearly the relevant Fermi volume that enters here is that of the up spins. Below we assume for simplicity that both the $\uparrow$ and $\downarrow$spins are at equal filling and so $\nu_\uparrow = \nu_\downarrow = \nu$ and $V_{FS}^{\uparrow} = V_{FS}^{\downarrow} = V_{FS}$. Equating the results from the trivial momentum counting, and the momentum counting above for the Fermi liquid (up to reciprocal lattice vector) yields

$$2\pi\nu L_y = \frac{2\pi}{L_x}\frac{V_{FS}}{(2\pi)^2} L_x L_y + 2\pi m_x,$$

$$2\pi\nu L_x = \frac{2\pi}{L_y}\frac{V_{FS}}{(2\pi)^2} L_x L_y + 2\pi m_y,$$

where m_x and m_y are integers and the two equations above are obtained from threading flux in the x and y directions. These quations can be rewritten as

$$N - L_x L_y \frac{V_{FS}}{(2\pi)^2} = L_x m_x, \tag{42}$$

$$N - L_x L_y \frac{V_{FS}}{(2\pi)^2} = L_y m_y, \tag{43}$$

where we have introduced the particle number $N = L_x L_y \nu$, an integer. In order to obtain the strongest constraint from these equations we consider a system with L_x, L_y mutually prime integers (no common factor apart from unity). Then, $m_x L_x = m_y L_y$ implies that they are multiples of $L_x L_y$; namely $m_x L_x = m_y L_y = p L_x L_y$ with p an integer. Thus we obtain the result

$$\nu = \frac{V_{FS}}{(2\pi)^2} + p, \tag{44}$$

which of course is Luttinger's theorem[3] that relates the Fermi volume to the filling (modulo filled bands that are represented by the integer p).

VIII. MOMENTUM BALANCE IN FL^*

Here we will consider an exotic variant of the Fermi liquid, where electron-like quasiparticles coexist with Z_2 fractionalization.[6] This state may be obtained beginning with a Z_2 fractionalized insulating state of electrons that breaks no lattice symmetries. We consider a specific model where the spinons ($f_\sigma^\dagger$ spin half, charge neutral excitations) are fermionic and the chargons ($b^\dagger$ spin zero, unit charged excitations) are bosonic. The electron operator is written as $c_\sigma^\dagger = b f_\sigma^\dagger$ and the relevant gauge structure is Z_2 which implies that pairing of spinons is present. If one is deep in the insulating phase then there is a large gap to the chargons; furthermore, if there is also a spin gap, then the low energy effective theory is just an Ising gauge theory. For an insulator with an odd number of electrons per site, in this regime we may set the chargon number $n_b = 0$ and the spinon number $n_f = 1$. The Ising gauge charge at each site is $(-1)^{n_b + n_f}$, which leads to an *odd* Ising gauge theory in this situation. For an insulator with even number of electrons per site, an *even* Ising gauge theory would result. We now imagine a situation where the lowest charge carrying excitation in the system is the electron itself. This could arise if the spinon and chargon form a tightly bound state so that it has a lower net energy than an isolated chargon. Doping would then lead to a "Fermi liquid" of electron-like quasiparticles, coexisting with gapped visons, spinons and chargons, which is the FL^* phase we wish to discuss. It already appears that a violation of Luttinger's relation may be expected here if we dope an insulator with an odd number of electrons per unit cell, since only the doped electrons may be expected to enter the Fermi volume. Here we will see how momentum balance arguments allows for such a violation, but nevertheless constrains the possible Fermi surface volumes so that a generalization of Luttinger's theorem to this exotic class of Fermi liquids holds.

In order to follow in detail the evolution of the system under flux threading we study the following model Hamiltonian:

$$H_{FL^*} = H_e^0 + H_{sp-ch}^0 + H_{int} + H_{gauge}, \tag{45}$$

$$H_e^0 = -\sum_{<ij>} t_{ij}^e c_{i\sigma}^\dagger c_{j\sigma}, \tag{46}$$

$$H_{sp-ch}^0 = -\sum_{<ij>} t_{ij}^c \sigma_{ij}^z b_i^\dagger b_j - \sum_{<ij>} t_{ij}^s \sigma_{ij}^z f_{i\sigma}^\dagger f_{i\sigma} + \Delta_s \sum_i (f_{i\uparrow}^\dagger f_{i\downarrow}^\dagger + h.c.), \tag{47}$$

$$H_{gauge} = -K \prod_{\Box} \sigma_{ij'}^z + h \sum \sigma_{ij}^x \tag{48}$$

with the constraint on all physical states

$$\prod_{j=nn(i)} \sigma_{ij}^x = (-1)^{n_b^i + n_f^i} \tag{49}$$

and H_{int} denotes the interactions between various fields which we do not specify here except for assuming that terms here do not couple to an externally imposed gauge field that is required for flux threading. Note, the spinons and chargons are coupled to the Z_2 gauge field, while the electrons are, of course, Z_2 gauge neutral. We have selected, for simplicity, an on site pairing interaction for the spinons; while such on site pairing terms are absent in microscopic models that forbid double occupancy of electrons, here we are concerned with universal aspects of quantum phases which are not affected by this simplification.

A. Flux threading in FL^*

We now consider the effect of flux threading on the ground state of the Hamiltonian (48). In the presence of a vector potential $A^{\uparrow}$ ($A^{\downarrow}$) coupling to the up (down) spin electrons, the hopping matrix elements for the up (down) spin electrons are modified to $t^e_{ij} \to t^e_{ij} e^{iA^{\uparrow}_{ij}}(t^e_{ij} \to t^e_{ij} e^{iA^{\downarrow}_{ij}})$, while the chargon hopping amlitude is modified to $t^c_{ij} \to t^c_{ij} e^{\frac{i}{2}(A^{\uparrow}_{ij}+A^{\downarrow}_{ij})}$ and the up (down) spinon hopping amplitude is modified to $t^s_{ij} \to t^s_{ij} e^{\frac{i}{2}(A^{\uparrow}_{ij}-A^{\downarrow}_{ij})}(t^s_{ij} \to t^s_{ij} e^{-\frac{i}{2}(A^{\uparrow}_{ij}-A^{\downarrow}_{ij})})$. Below we imagine threading 2π flux in $A^{\uparrow}$ and study the evolution of the ground state of the system in this process. In addition to the excitation of particle-hole pairs of the electron-like Fermi-liquid quasiparticles, we will also see that in some situations a vison excitation is inserted through the cylinder which gives rise to the modified Luttinger relations. We begin by considering flux threading in the absence of vison dynamics [$h=0$ in Eq. (48)], where results are easily derived, and then reinstate the vison dynamics and show that the central result is unaffected.

The adiabatic insertion of a unit flux quantum that couples to the up spin electrons is affected by introducing a gauge field on the horizontal links of the cylinder in Fig. 1 and increasing its strength from zero ($A^{\uparrow}_{ij}=0 \to 2\pi/L_x$) in time T. The time evolution of the quantum state can be written as $|\psi(T)\rangle = \mathcal{U}_T|\psi(0)\rangle$ where $\mathcal{U}_T = \mathcal{T}_t \exp(-i\int_0^T H_{FL^*}(t)dt)$ where $\mathcal{T}_t$ is the time-ordering operator, and the time dependence of the Hamiltonian arises from the flux threading. Clearly, since 2π of flux is invisible to the electrons, the final state must be some excited state of the initial ($A^{\uparrow}=0$) Hamiltonian. In order to make this explicit, the 2π flux is gauged away, which can be accomplished by the operators $\mathcal{U}_\sigma \mathcal{U}_\phi$ with

$$U_\phi = \exp\left\{ i\frac{2\pi}{L_x}\sum_i x_i\left(n^e_{\uparrow i} + \frac{1}{2}\left(n^i_{f\uparrow} - n^i_{f\downarrow} + n^i_b\right)\right)\right\},$$

$$\mathcal{U}_\sigma = \prod_{ij\in \mathrm{cut}} \sigma^x_{ij}. \qquad (50)$$

While the first unitary operator eliminates the gauge field for the electrons, it changes the sign of the hopping matrix element on a single column of horizontal links for the chargons and spinons which behave like half charges. This modification to the hopping can be absorbed in the Z_2 gauge fields, which is accomplished by the unitary operator $\mathcal{U}_\sigma$, which returns us to the initial Hamiltonian.

The action of the time evolution operator $\mathcal{U}_T$ is to excite electron-like quasiparticles about the Fermi surface in the usual manner, while the gapped spinons and chargons are not excited during this adiabatic flux threading. In the absence of vison dynamics [$h=0$ in Equation (48)], it may be seen that a vison is also introduced during the flux threading proceedure. This is argued as follows. In the absence of vison dynamics, the vison number through the hole of the cylinder, as measured by the operator $W_C = \Pi_C \sigma^z_{ij}$, where C is a contour that winds around the cylinder, is a good quantum number since it commutes with the Hamiltonian in this limit. However, in the course of flux threading and returning to the original gauge, it changes sign since it may be easily verified that $\mathcal{U}_\sigma W_C \mathcal{U}_\sigma^{-1} = -W_C$, which implies vison insertion. Thus, the final state has a displaced Fermi sea and a vison.

We can now combine the results of trivial momentum counting and a knowledge of the vison momentum to obtain the volume of electron-like quasiparticles. This is most easily done in the limit of a very large gap to the gauge charged particles (spinons and chargons). Then, the phase is described by gauge neutral electrons forming a Fermi liquid and a pure Ising gauge theory in the deconfined phase. We know that the vison excitations of the latter through the hole of the cylinder with an odd number of rows carries crystal momentum 0 or π depending on the even or odd nature of the Ising gauge theory. The gauge constraint in Eq. (49) tells us this depends on the parity of $n_f + n_b$ at each site. If we set $n_b=0$ for the gapped chargons and $n_f=1$ for the gapped spinons, where for the latter we assume the system is obtained continuously by doping a spin liquid with spin 1/2 per unit cell (i.e., a spin version of I^*_{odd}). In this limit an odd Ising gauge theory will be obtained, where the vison threading an odd width cylinder carries crystal momentum π. If the gap to the spinons and chargons is now reduced from infinity, this crystal momentum assignment to the vison cannot change continuously (from time reversal symmetry), and is hence expected to be invariant for a finite range of gap values. Thus, the phase is expected to be continuously connected to the large gap situation with integer or half integer filling, which will determine the momentum assignments.

Thus, we are left with the result that two types of exotic Fermi liquid states FL^*_{even} and FL^*_{odd} are expected, that differ in the crystal momentum carried by the vison excitations. The momentum balance argument then immediately implies that these two states will have different Fermi volumes at the same filling—while FL^*_{even} will have a Fermi volume that is identical to that of a conventional Fermi liquid at the same filling and hence respects Luttinger's relation, FL^*_{odd} has a Fermi volume that violates Luttinger's relation in a very definite way.

Since the only situation where the momentum balance argument will give a result distinct from that of a conventional Fermi liquid is for the case of FL^*_{odd} on an even×odd lattice, where the vison carries a nontrivial crystal momentum, we discuss that below. Consider flux threading in such a phase on a cylinder with an odd number of rows. This will introduce a vison through the hole of a cylinder which carries crystal momentum π. This needs to be subtracted from the usual momentum balance relations for a Fermi liquid displayed in Eq. (43). This leads to a modified Luttinger relation between the Fermi volume in FL^*_{odd} and the electron filling ν

$$\{\nu - \frac{1}{2}\} - p = \frac{V_{FS}^*}{(2\pi)^2}, \tag{51}$$

where p is an integer that represents filled bands. We reiterate that ν is the number of electrons of each spin per unit cell. The crucial difference from the usual Luttinger relation in Eq. (44) is the fact that the Fermi volume is determined by $\nu - \frac{1}{2}$, which is related to the fact that it is obtained by doping the fractionalized spin model which is translationally symmetric at half filling.

Finally, we argue that reintroducing the vison hopping $(h \neq 0)$ does not affect these conclusions. In the cases where the vison threading the cylinder carries zero crystal momentum, introducing vison hopping leads to a mixing of the vison and no vison states in a finite system. This implies that we are no longer guaranteed to have a vison on flux threading. However, for the case of FL_{odd}^* on a cylinder with an odd number of rows, where the vison carries crystal momentum π, the nontrivial crystal momentum blocks the tunneling of visons even on a finite sized system. This implies that flux threading does indeed introduce a vison which finally leads to the modified Luttinger relation. We can see that the π crystal momentum carried by the vison cannot be transferred to the only other gapless excitations in the problem, the electron-like quasiparticles, since they are gauge neutral.

Our argument is an expanded version of the basic ideas about FL_{odd}^* noted in Ref. 6. However, our proof of the modified Luttinger relation for FL_{odd}^* is more comprehensive, and we also identify the FL_{even}^* phase which is an exotic Fermi liquid but nevertheless obeys the conventional Luttinger relation.

IX. CONVENTIONAL SUPERFLUID $\mathcal{SF}$

Luttinger's theorem was formulated for Fermi liquids, and we have extended Oshikawa's argument to show how the theorem must be modified to account for the existence of gapped spin liquid insulators (which may be equivalently viewed as fractionalized bosonic insulators) as well as fractionalized Fermi liquids. In both cases, the presence of topological order was crucial. Let us next turn to conventional superfluids and ask: What property of the superfluid phase is captured by the Oshikawa argument, and gets fixed by the particle density? While we focus on the case of bosonic superfluids, we expect our results to also be applicable to s-wave superconductors with a large gap, so that the resulting Cooper pairs may be effectively viewed as bosons. Also, we consider the case of neutral bosons (no internal electromagnetic gauge field). Again, these results could be applied approximately to the case of charged superfluids if the penetration depth is sufficiently large.

Conventional superfluids in two or more dimensions ($D \geqslant 2$) are Bose condensed at zero temperature, and have a unique ground state (on both cylinders and torii). It is clear that the Oshikawa argument must then capture some property of the excitations in the superfluid. A conventional superfluid supports two kinds of excitations: the gapless linearly dispersing Goldstone mode of the broken symmetry state ("phonon"), and topological defects, namely vortices. We show below that it is the Berry phase acquired by a vortex on adiabatically going around a closed loop that is fixed by the particle density ν, independent of the strength and nature of interactions between bosons in the superfluid.

A. Effective Hamiltonian for $\mathcal{SF}$

We may describe a conventional superfluid most conveniently in a rotor representation for the bosons—thus $B_{\mathbf{r}}^\dagger \to e^{-i\phi_{\mathbf{r}}}$, $B_{\mathbf{r}}^\dagger B_{\mathbf{r}} \to N_{\mathbf{r}}$ with $[\exp(i\phi_{\mathbf{r}}), N_{\mathbf{r}'}] = \exp(i\phi_{\mathbf{r}})\delta_{\mathbf{rr}'}$, and the Hamiltonian for interacting bosons in these variables takes the form

$$\hat{H}_A(\mathcal{SF}) = -t_b \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j + QA_{ij}) + V_{\text{int}}[n], \tag{52}$$

where the interactions may be of the general form $V_{\text{int}}[n] = \Sigma_{\mathbf{rr}'} U_{\mathbf{rr}'} N_{\mathbf{r}} N_{\mathbf{r}'}$. The Bose condensed superfluid, which results in dimensions $D \geqslant 2$ when t_b is the largest scale in the Hamiltonian, supports linearly dispersing phonons, which are the Goldstone mode of the broken symmetry. Vortices appear as topological defects in the phase field in the ordered state, and there is a nonzero gap to creating vortices in the bulk of the superfluid.

In the above discussion, we have assumed that the phase variable has periodic boundary conditions, namely, $\varphi_{\mathbf{r}+L_x} = \varphi_{\mathbf{r}}$, $\varphi_{\mathbf{r}+L_y} = \varphi_{\mathbf{r}}$. However, on cylinders/torii the superfluid has additional excited states corresponding to creating vortices through holes of the cylinders/torii. A state with $\varphi_{\mathbf{r}+L_x} = \varphi_{\mathbf{r}} + 2\pi m_x$ corresponds to a strength m_x vortex through a hole in the cylinder.

B. Flux threading in $\mathcal{SF}$

Consider N bosons each with charge Q condensed into a conventional superfluid ground state on an $L_x \times L_y$ lattice in the form of a torus. Trivial momentum counting tells us that threading flux $\Phi_0 = hc/Q$ into the cylinder on which the system lives changes the crystal momentum by $2\pi\nu L_y$, where ν is the filling and L_y is the number of rows of the cylinder. We show below, using a low energy description of the superfluid, that adiabatic flux threading introduces a vortex into the hole of the cylinder/torus. We do this in two steps. First, we turn off the boson interactions which allows us to directly construct the final state and see that it corresponds to introducing one vortex. Next, we turn back on the boson interactions and argue that this does not affect the state or change the momentum carried by the vortex.

1. Threading flux Φ_0 introduces a vortex

Let us adiabatically thread flux hc/Q in the $-\hat{y}$ direction for the above system on a cylinder as in Fig. 1 such that starting from the initial eigenstate $|\Psi(0)\rangle$ in the absence of flux, the final state reached is $|\Psi(T)\rangle$. The final state can, of course, be written as $|\Psi(T)\rangle = \mathcal{U}_T|\Psi(0)\rangle$ with $\mathcal{U}_T = \mathcal{T}_t \times \exp(-i\int_0^T H_A(\mathcal{SF}, t)dt)$ where $\mathcal{T}_t$ is the time-ordering operator. We can go to the $A_{ij}=0$ gauge by making a unitary transformation $H_A(\mathcal{SF}, T) \to \mathcal{U}_G H_A(\mathcal{SF}, T)\mathcal{U}_G^{-1} \equiv H_0(\mathcal{SF})$ (corresponding to zero vector potential). Here $\mathcal{U}_G = \mathcal{U}_\phi$, with

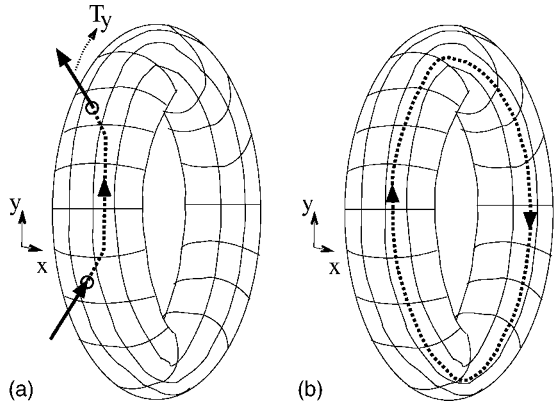

FIG. 4. Threading a vortex through the hole of a torus. (a) A vortex (arrow coming out) – antivortex (arrow going in) pair is created, and separated by the translation operator T_y. (b) When the vortex is taken all the way around the torus and then annihilated with the antivortex, the resulting state has one unit of circulation about the hole of the torus as shown.

$$\mathcal{U}_\phi = \exp\left(i\frac{2\pi}{L_x}\sum_i x_i\hat{n}_i \right). \tag{53}$$

Thus, the final state in the $A_{ij}=0$ gauge is $|\Psi_f\rangle = \mathcal{U}_\phi \mathcal{U}_T |\Psi_i\rangle$.

Let us consider the extreme limit achieved by $V_{\rm int}\to 0$. In this case, the system is initially in the full Bose condensed state $|\exp(i\phi_i)=1\rangle$, which is unaffected by the time development operator $\mathcal{U}_T$, and the final state after acting with $\mathcal{U}_\phi$ has $|\exp(i\phi_i)=\exp(i2\pi x_i/L_x)\rangle$, namely it corresponds to having a single vortex threading the hole of the cylinder. Trivial momentum counting then tells us that this vortex carries momentum: $P_{\rm vortex}=2\pi\nu L_y$.

2. Flux threading in the presence of boson interactions

In the presence of boson interactions, the phase φ at each site is no longer a c number. Since the number and phase do not commute, interactions introduce phase fluctuations. Such fluctuations may permit the vortex to escape from the hole of the cylinder. Clearly, this is only possible if the initial and final state have the same crystal momentum quantum number. Since the vortex state carries $P_{\rm vortex}=2\pi\nu L_y$, we expect the vortex will remain trapped except at special values of ν and L_y, where $P_{\rm vortex}$ becomes a multiple of 2π. Thus, in general, even in the presence of boson interactions, threading flux Φ_0 introduces one vortex, carrying the above crystal momentum, into the hole of the cylinder.

3. Flux threading and the Berry phase for a vortex

It is well known that moving vortices in a stationary Galilean invariant superfluid experience the so-called Magnus force, a force which acts transverse to the velocity of the vortex. A superfluid vortex thus behaves as a charged particle in a magnetic field, the Magnus force being analogous to the transverse Lorentz force. In a lattice system, this "magnetic field" seen by the vortex is encapsulated through vector potentials living on the links of the lattice, and the vortices pick up a Berry phase χ (of the Aharonov-Bohm kind) on going around an elementary plaquette of the lattice. We will now show that using the momentum balance argument fixes this Berry phase to be $\chi=2\pi\nu$.

Consider a single vortex on an infinite plane. Since the vortex sees a "flux" χ per plaquette of the lattice, the unit translation operators for the vortex satisfy: $T_xT_y=T_yT_x \times\exp(-i\chi)$. Let $|K_X,Y\rangle$ represent the state with one vortex with x-crystal momentum K_X located at $y=Y$. When this vortex is translated by one unit along the $+\hat{y}$ direction, i.e., to $y=Y+1$ it is straightforward to show that the new state has x-crystal momentum given by $K_x+\chi$. For an antivortex, the translation operators satisfy $T_xT_y=T_yT_x\exp(i\chi)$, and translating the antivorton along $+\hat{y}$ changes the crystal momentum to $K_x-\chi$.

With this in mind, let us thread a vortex through the torus in the manner shown in Fig. 4. Start with a state with well-defined crystal momentum along the x direction, say zero. Create a vortex-antivortex pair on some plaquette and make a superposition with *zero* net crystal momentum along $\hat{x}$. Next drag them apart by translating the vortex along $+\hat{y}$ using the translation operator T_y until they are L lattice spacings apart along the torus. This state then has additional crystal momentum χL. If we drag the pair all the way around the torus and annihilate the vortex-antivortex pair, this would be equivalent to threading a vortex through the torus as in Fig. 4(b). The net momentum change is then χL_y. On the other hand, we have shown that $P_f-P_i=2\pi\nu L_y$ for threading the vortex through the hole of the torus. This fixes $\chi=2\pi\nu$.

We have confirmed this result by using the well known duality mapping[28] between bosons and vortices in 2+1 dimensions. The dual theory treats vortices as point particles minimally coupled to a noncompact U(1) gauge field. In the dual theory, the flux of the gauge field on an elementary plaquette as seen by the vortex emerges naturally as $\chi=2\pi\nu$, i.e., the vortices see each boson as a source of 2π magnetic flux. The noncompactness of the gauge field, or the conservation of the magnetic flux piercing the lattice, is a simple consequence of total boson number conservation.

To summarize, vortices in a uniform superfluid pick up a Berry phase $\chi\mathcal{N}$ on adiabatically going around a loop enclosing $\mathcal{N}$ plaquettes. The Berry phase per plaquette is completely determinted by the particle density as The Berry phase per plaquette is completely determined by the particle density as $\chi=2\pi\nu$. Writing $\chi=2\pi\alpha_M$, which defines the "Magnus coefficient" α_M, leads to the Luttinger relation for superfluids, namely

$$\nu = \alpha_M + p. \tag{54}$$

This relation follows from using the momentum balance arguments of Oshikawa, applied to a conventional superfluid. In this sense, the Berry phase relation above may be viewed as the analogue of the Luttinger relation for Fermi liquids.

There is a concern which we have not addressed so far—the vortex may have a modified density near its core, and this in turn could modify the Berry phase accumulated by the vortex when it is adiabatically taken around a loop. However, this term does not change with the area of the loop, so we can still use the above result to deduce a precise difference of the Berry phase between two loops enclosing different areas.

We need to define $2\pi\alpha_M \equiv \Delta\Phi/\Delta\mathcal{N}$, where $\Delta\Phi$ is the difference in Berry phase between two loops which differ in area by $\Delta\mathcal{N}$ plaquettes. Defined in this manner, $2\pi\alpha_M$ obeys the precise relation in Eq. (54). Another caveat is the very definition of adiabaticity in the presence of gapless superfluid phonon excitations—to be completely rigorous, we need to work with a finite-sized system such that the phonons have a nonzero gap, and demand that the vortex motion be adiabatic with respect to this energy scale.

X. Z_2 FRACTIONALIZED SUPERFLUID $\mathcal{SF}^*$

We now discuss an exotic variant of the superfluid, $\mathcal{SF}^*$, and see how the relation (54) is modified in this phase. The $\mathcal{SF}^*$ phase is a Z_2 fractionalized superfluid which was first discussed by Senthil and Fisher.[9] It supports three distinct gapped excitations: (i) an elementary hc/Q vortex (called the "vorton" in Ref. 9), (ii) Z_2 gauge flux (the "vison"), and (iii) an electromagnetically neutral particle carrying Z_2 gauge charge (the "ison"). There are various ways in which the $\mathcal{SF}^*$ phase may be realized; we shall briefly outline one of them.

Let us start with a fractionalized insulator $\mathcal{I}^*$ which can be realized at integer or half odd integer density of bosons. This supports two gapped excitations: charge $Q/2$ chargons that also carry Z_2 gauge charge and Ising vortices (visons). On doping this insulator, the additional charge-Q bosons can deconfine into pairs of chargons since in the fractionalized phase Bose condensing the doped chargons would destroy deconfinement of the Z_2 gauge field (by the Anderson-Higgs mechanism, since the condensate carries Z_2 gauge charge) and lead to á conventional superfluid.

The other possibility is that doped chargons *pair* and Bose condense resulting in a superfluid phase. Since the condensate is Z_2 gauge neutral, deconfinement is preserved and this exotic superfluid phase is called $\mathcal{SF}^*$. It supports elementary hc/Q vortices, and the visons still survive in the superfluid. There is, however, another excitation, analogous to a Bogoliubov quasiparticle of a superconductor, present in the system—this gapped quasiparticle[9] is the ison. It may be viewed as a descendant of the chargon in the insulator, whose electric charge has been screened by the Bose condensate of pairs so that it only carries a Z_2 Ising charge.[29] In addition to these gapped excitations, there is, of course, the gapless superfluid phonon in an electrically neutral system. The relative statistics of the gapped excitations are as follows: the wave function changes sign if an ison is adiabatically taken around the vison or the vorton.

We have seen how Z_2 fractionalized Bose Mott insulators with full lattice translation symmetry fall into two classes, depending on whether the boson filling is an integer ($\mathcal{I}^*_{even}$) or half odd integer ($\mathcal{I}^*_{odd}$) as described in Sec. V. Similarly, fractionalized Fermi liquids $\mathcal{FL}^*$ also come in two varieties as shown before, with different relations between fermion filling and Fermi volumes. It is not surprising, therefore, that we will find below two kinds of Z_2 fractionalized superfluids, $\mathcal{SF}^*$. Using momentum counting arguments as done earlier, we will also see how the distinction between the two types of $\mathcal{SF}^*$ phases, namely $\mathcal{SF}^*_{even}$ and $\mathcal{SF}^*_{odd}$, is reflected in different dynamics for the vorton in these two cases. While in the case of translationally symmetric Z_2 insulators $\mathcal{I}^*$, a knowledge of the filling alone was enough to determine the odd/even nature of the phase, this is no longer true in the case of Z_2 fractionalized superfluids, $\mathcal{SF}^*$ (or $\mathcal{FL}^*$), where additional information regarding the odd/even nature of the phase is needed.

A. Effective Hamiltonian for $\mathcal{SF}^*$

A simple description of the $\mathcal{SF}^*$ phase may be obtained by using a Hamiltonian which describes chargons minimally coupled to a Z_2 gauge field. This takes the form

$$H(\mathcal{SF}^*) = \hat{T} + \hat{V} + H_g \tag{55}$$

with

$$\hat{T} = -t_b \sum_{\langle ij\rangle} \sigma^z_{ij} \cos(\phi_i - \phi_j + QA_{ij}/2) - t_B \sum_{\langle ij\rangle} \cos(2\phi_i - 2\phi_j + QA_{ij}),$$

$$\hat{V} = v_{\rm int}[n],$$

$$H_g = -K \sum_{\square} \prod_{\square} \sigma^z_{ij} - h \sum_{\langle ij\rangle} \sigma^x_{ij} \tag{56}$$

and physical states of the theory satisfy the constraint in Eq. (21). Here $\exp(-i\phi_i)$ creates a chargon carrying charge $Q/2$ and Z_2 gauge charge at site i, n_i is the chargon number, and the terms in $\hat{T}$ represent the chargon kinetic energy. Single chargons hop with an amplitude t_b, and are coupled minimally to the Z_2 gauge field and for simplicity of discussion, we have included explicitly a chargon-pair hopping term with amplitude t_B. Clearly a chargon-pair created by $\exp(-2i\phi_i)$ has no net Ising charge and does not couple to the Z_2 gauge field. $V_{\rm int}[n]$ is an interaction term involving chargon densities which we do not spell out here. The exotic superfluid phase $\mathcal{SF}^*$ requires being in the deconfined phase of the gauge theory (which is guaranteed by a large K/h) and with chargon pairs condensed (which can be achieved with large chargon pair hopping t_B, while single chargon hopping remains small).

Let us now write down the effective Hamiltonian in the $\mathcal{SF}^*$ phase. Condensation of chargon pairs implies that we can replace the operator $\exp(-2i\phi_i)$ by a c number. Then, the magnitude of the chargon creation operator is determined, but its sign can fluctuate which gives rise to the ison field, i.e., we can write $\exp(-i\phi_i) \propto I^z_i$, where I^z_i is a Pauli matrix with eigenvalues ± 1. Similarly, since chargon pairs are condensed, the parity of the chargon number operator chargon number n_i must be changed by the ison creation operator I^z_i, hence we identify $n_i \approx (1+I^x_i)/2$; again I^x is a Pauli matrix and $(1+I^x)/2$, with eigenvalues $\{0,1\}$, counts the number of unpaired Ising charged particles.

In new variables, the Hamiltonian (with $A_{ij}=0$) reduces to

$$H_{\rm red} = \hat{T}_{\rm red} + \hat{V}_{\rm red} + H_g + H_{\rm condensate}, \tag{57}$$

$$\hat{T}_{\rm red} = -t'_b \sum_{\langle ij\rangle} I_i^z I_j^z \sigma_{ij}^z,$$

$$\hat{V}_{\rm red} = -g\sum_i I_i^x, \tag{58}$$

where we have introduced a "chemical potential" g for the isons, and $H_{\rm condensate}$ describes the dynamics of the condensate. Physical states of this theory need to satisfy the constraint

$$\prod_{j=nn(i)} \sigma_{ij}^x = -I_i^x. \tag{59}$$

This is simply an Ising model coupled to an Ising gauge field—the $\mathcal{SF}^*$ phase is realized when both the isons (excitations of the Ising model) visions (excitations of the Ising gauge theory) are gapped. This will occur when K/h and $|g/t'_b|$ are large, when the gauge theory is in the deconfined phase and the Ising model is "disordered." Let us now briefly consider the special limiting cases where $|g|\to\infty$ in order to expose the underlying reason for the two kinds of Z_2 fractionalized $\mathcal{SF}^*$ phases. Clearly, if $g\to\pm\infty$, we would have $I_i^x=\pm 1$ corresponding to the ison number $(1+I_i^x)/2=0$ respectively. The physical states of the gauge theory then satisfy the constraint

$$\underline{g\to\mp\infty}: \prod_{j=nn(i)} \sigma_{ij}^x = \pm 1. \tag{60}$$

These constraints on the gauge theory, as we know from the discussion on insulators, correspond to even and odd Ising gauge theories, respectively, which correspond to having zero or one Ising charged particle (ison) fixed at each site. Thus, for $g\to+\infty$ and for $g\to-\infty$ we will obtain two distinct superfluid phases, which we label $\mathcal{SF}^*_{even}$ and $\mathcal{SF}^*_{odd}$, respectively, which persist to finite values of g as well. These are separated by an intermediate phase where $|g|\ll t_b$ where the I^z Ising field orders; this is the conventional superfluid phase. Below, we will see how these $\mathcal{SF}^*$ phases can be distinguished from each other.

B. Flux threading in $\mathcal{SF}^*$

Trivial momentum counting tells us that threading flux $\Phi_0=hc/Q$ into the cylinder on which the system lives changes the crystal momentum by $2\pi\nu L_y$, where ν is the filling and L_y is the number of rows of the cylinder. Where is this momentum soaked up in the $\mathcal{SF}^*$ phase? We will show below that flux threading introduces both a vison and a vorton into the hole of the cylinder. The crystal momentum is then divided up between these two excitations, in a way that depends on whether we are dealing with $\mathcal{SF}^*_{even}$ or $\mathcal{SF}^*_{odd}$. This will be argued below in two stages. First, we consider freezing the Ising gauge field dynamics by setting the vison hopping to zero ($t=0$). There it can easily be argued that Φ_0 flux threading leads to both a vison and a vorton. Then, using our earlier knowledge of vison momenta in the even/odd Z_2 gauge theories, and the total momentum imparted to the system, we can read off the crystal momentum carried by the vorton. Finally, we reinstate the vison hopping ($t>0$) and use continuity to argue that this does not affect the momentum assignments.

1. Threading flux Φ_0 introduces a vison and vorton:

Consider at first the limit $h=0$ and $t_B=\infty$ identically, so that before introducing flux we can everywhere set $\sigma_{ij}^z=1$ (as a reference state which we can then project into the subspace of physical states) and $\exp(i2\phi_i)=1$.

Let us adiabatically thread flux hc/Q in the $-\hat{y}$ direction for the above system on a cylinder as in Fig. 1 such that starting from the initial eigenstate $|\Psi(0)\rangle$ in the absence of flux, the final state reached is $|\Psi(T)\rangle$. The final state can, of course, be written as $|\Psi(T)\rangle=\mathcal{U}_T|\Psi(0)\rangle$ with $\mathcal{U}_T=\mathcal{T}_t \times\exp(-i\int_0^T H_A(\mathcal{SF}^*,t)dt)$ where $\mathcal{T}_t$ is the time-ordering operator. We can go to the $A_{ij}=0$ gauge by making a unitary transformation $H_A(\mathcal{SF}^*,T)\to\mathcal{U}_G H_A(\mathcal{SF}^*,T)\mathcal{U}_G^{-1}\equiv H_0(\mathcal{SF}^*)$ (corresponding to zero vector potential). Here $\mathcal{U}_G=\mathcal{U}_\phi\mathcal{U}_\sigma$, with

$$\mathcal{U}_\phi = \exp\left(i\frac{\pi}{L_x}\sum_i x_i\hat{n}_i\right),$$

$$\mathcal{U}_\sigma = \prod_{ij\in{\rm cut}} \sigma_{ij}^x \tag{61}$$

and cut refers to the vertical column of links for which $x_i=L_x$, $x_j=1$ [shown in Fig. 3(b)]. Thus, the final state in the $A_{ij}=0$ gauge is $|\Psi_f\rangle=\mathcal{U}_\phi\mathcal{U}_\sigma\mathcal{U}_T|\Psi_i\rangle$. Since the system is a superfluid initially in the state $|\exp(2i\phi_i)=1\rangle$, which is unaffected by the time development operator $\mathcal{U}_T$ since we are in the $t_B=\infty$ limit, the final state after acting with $\mathcal{U}_\phi$ has $|\exp(2i\phi_i)=\exp(i2\pi x_i/L_x)\rangle$, namely it corresponds to having a single vorton threading the hole of the cylinder.

At the same time, following arguments similar to the insulator $\mathcal{I}^*$, acting with $\mathcal{U}_\sigma$ introduces a vison in the hole, the vison number being defined by $(1-W_C)/2$ with $W_C=\Pi_C\sigma_{ij}^z$ where the loop C is taken around the cylinder [see Fig. 3(b)].

Thus, for $h=0$, $t_B=\infty$, threading a 2π flux adds a vison and a vorton into the hole of the cylinder. Thus, when the effective description of the gauge fields is an even (odd) Ising gauge theory (which are connected to the $g\to-\infty(+\infty)$ limits, respectively, as we have seen above), momentum counting arguments already showed us that the vison carries momentum zero (for the even gauge theory) or πL_y (for the odd gauge theory). The remaining momentum must clearly be carried by the vorton!

Thus, in the case of even Z_2 gauge theories the vison carries zero crystal momentum, and we deduce that the vorton through the hole of the cylinder carries crystal momentum $2\pi\nu L_y$, and this is the $\mathcal{SF}^*_{even}$ case. For the case of odd Z_2 gauge theories, the vison carries crystal momentum πL_y and the vorton carries crystal momentum $2\pi(\nu-1/2)L_y$, and this is the $\mathcal{SF}^*_{odd}$ case. To summarize, momentum balance arguments suggest

$$P^{even}_{\rm vorton} = 2\pi\nu L_y[{\rm mod}\ 2\pi], \tag{62}$$

$$P^{odd}_{\text{vorton}} = 2\pi\left(\nu - \frac{1}{2}\right)L_y[\text{mod } 2\pi]. \tag{63}$$

We now show that these momentum assignments are not affected on reinstating the vison hopping ($h \neq 0$) and the condensate dynamics ($t_B \neq \infty$).

2. Flux threading with dynamical gauge fields

Here, starting from the case with $h=0$, $t_B=\infty$, let us ask what happens if we turn on a nonzero h and finite t_B, giving dynamics to the gauge field and to the condensate.

In this case the loop product W_C no longer commutes with the Hamiltonian, and we cannot use its eigenvalues to label the eigenstates. Thus, in a finite system as argued previously for the case of inulators, the vison can tunnel out in the case of even gauge theories for any cylinder dimension, or in the odd gauge theories, if the cylinder has an even number of rows (L_y). However, for the case of odd gauge theories on a cylinder with odd number of rows (L_y), the vison carried momentum π, and hence vison tunneling is blocked even in finite systems. Similarly, when there is dynamics to the condensate ($t_B \neq \infty$), the vorticity is also not necessarily conserved in a finite system, namely $\int \nabla 2\phi$ around the cylinder is not a constant (not a classical variable). More precisely, this statement can be made in terms of an order of limits; if the time scale for flux threading is taken to infinity before the thermodynamic limit is taken, then the system can remain in the zero vorticity state at the end of the flux threading. This occurs if the two states, namely the state with no vison and no vorton and the state with 1 vison and 1 vorton, each carry zero crystal momentum, and can then mix to give eigenstates of the Hamiltonian. This happens if $2\pi\nu L_y = 0(\text{mod } 2\pi)$. Otherwise, even in the presence of h and condensate dynamics, the vorton acquires a nonzero crystal momentum and therefore its tunneling is blocked even in a finite size system (in the sense described above). Thus, in all cases where a nontrivial crystal momentum is imparted to the system from flux threading, this is accounted for by the presence of a vorton and/or a vison in the final state which carries the appropriate momentum.

The two superfluids $\mathcal{SF}^*_{even}$, $\mathcal{SF}^*_{odd}$ can then be distinguished depending on the momentum carried separately by the vison and the vorton though the total momentum carried by these excitations is the same. We shall later see that this may be reflected in the measured Hall effect in the vorton liquid phase in these systems through the Berry phase induced Magnus force on the vorton. In the next subsection, we shall directly verify the momentum assigments for the vorton by going to a dual theory where vortices are represented as particles.

C. Berry phase for vorton and consistency with momentum counting

The dual theory for the $\mathcal{SF}^*$ phase, where the chargons are traded for vortex variables, is derived in Appendix A. The action takes the form

$$S = S_{gauge} + S_c,$$

$$S_{gauge} = -\varepsilon K \prod_{\square_s} \sigma_{ij} - \varepsilon K_\tau \prod_{\square\tau} \sigma_{ij},$$

$$S_c = -\varepsilon t_v \sum_{i,\mu} \cos(\theta_i - \theta_{i+x_\mu} - 2\pi\mathcal{A}^i_\mu - \pi a^i_\mu) - \varepsilon g \sum_i I^i_0 + i\frac{\pi}{2}\sum_{i,\mu} I^i_\mu(1-\sigma_{i,i+\tau}) + \alpha \sum_{i,\mu=1,2} (\mathcal{J}^i_\mu)^2 + \varepsilon U \sum_i (\mathcal{J}^i_0 - \bar{n})^2. \tag{64}$$

Here $\exp(-i\theta)$ creates a vorton, and the first term in S_c represents vorton hopping. The vortons appear as charged particles minimally coupled to the gauge fields $\mathcal{A}_\mu$ and a_μ. The flux of the (noncompact) $U(1)$ gauge field $\mathcal{A}_\mu$ and the gauge field a_μ are tied to the charge current ($\mathcal{J}_\mu$) and the ison current (I_μ), respectively

$$\mathcal{J}^i_\mu/2 = \epsilon_{\mu\nu\lambda}\partial_\nu \mathcal{A}^i_\lambda, \tag{65}$$

$$I^i_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu a^i_\lambda. \tag{66}$$

The isons have a chemical potential g, and a Berry phase term associated with the fact that they couple to the Z_2 gauge field σ_{ij}. The remaining terms represent local interactions between the charge density/currents.

On the spatial links ($\mu=1, 2$), $\langle I^i_\mu\rangle=0$ and $\langle \mathcal{J}^i_\mu\rangle=0$. On the temporal links, for the chargon density we have $\langle \mathcal{J}^i_0\rangle = 2\nu$, where ν is the average charge density in units of Q (equivalently, ν is the density of chargon *pairs*). For the ison density, we have two possibilities: (i) for $g \to -\infty$, we have $I^i_0=0$ and there are no isons in the ground state; (ii) for $g \to +\infty$, there is one ison nailed down to each lattice site, $I^i_0=1$. For $g \to -\infty$ (or $\mathcal{SF}^*_{even}$), when the vortons go anticlockwise around an elementary plaquette of the square lattice, they see only the flux produced by the vector potential $2\pi\mathcal{A}$, and the wave function acquires a factor $\exp(i\pi\mathcal{J}^i_0)$. On average, the phase picked up is thus $2\pi\nu$. This is identical to the Berry phase picked up by a vortex in a conventional superfluid with boson density ν. For $g \to +\infty$ (or $\mathcal{SF}^*_{odd}$), the vorton sees an *additional* flux produced by one ison charge $I^i_0=1$ nailed down to each site, and the total flux seen by the vorton is thus $2\pi(\nu-1/2)$—this deviates by π from the Berry phase for $\mathcal{SF}^*_{even}$ and the conventional superfluid.

Applying the picture of vortex threading presented for conventional superfluids to this case of vorton threading, it is clear that this Berry phase is consistent with the momentum counting argument. Namely, the vorton threading suggests that $\chi L_y = P_{\text{vorton}}$. On the other hand, momentum balance tells us $P_f - P_i = 2\pi\nu L_y$ for threading *a vison and a vorton* through the hole of the torus. Since we know that the vison carries crystal momentum $P^{(even)}_{\text{vision}}=0$ in an even gauge theory (as in $\mathcal{SF}^*_{even}$) or $P^{(odd)}_{\text{vision}}=\pi L_y$ in an odd gauge theory (as in $\mathcal{SF}^*_{odd}$). This fixes P_{vorton} in the two cases [in agreement with Eqs. (62) and (63)] and thus $\chi^{(even)}=2\pi\nu$, $\mathcal{SF}^*_{even}$) and $\chi^{(odd)} = 2\pi(\nu-1/2)$ (in $\mathcal{SF}^*_{odd}$). This is consistent with the result derived from the dual theory above. Defining as before the Magnus coefficient $\alpha_M = \chi/2\pi$, we obtain

$$\alpha^{even}_M = \nu - p, \tag{67}$$

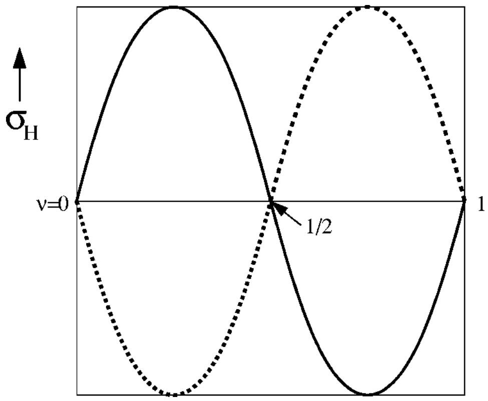

FIG. 5. Schematic figure showing the filling dependence [$\nu=0$ (1) is the empty (full) band] of the Hall conductivity in a conventional Fermi liquid or $\mathcal{FL}^{*}_{even}$ (solid line), for electrons within a simple Drude-like picture, contrasted with the behavior expected in the $\mathcal{FL}^{*}_{odd}$ phase (dashed line). Similar results are expected for the conventional superfluid or $\mathcal{SF}^{*}_{even}$ phase compared to the $\mathcal{SF}^{*}_{odd}$ phase in a vortex liquid phase.

$$\alpha_M^{odd} = \left\{ \nu - \frac{1}{2} \right\} - p, \tag{68}$$

where p is an arbitrary integer. Thus the momentum counting provides a prescription to fix the Berry phase for the vorton, and allows us to distinguish the odd and even exotic superfluid phases.

XI. CONCLUSIONS

Extending a nonperturbative argument, made by Oshikawa for the Fermi liquid, we have constructed analogues of Luttinger's theorem for systems other than the conventional Fermi liquid in dimensions $D \geqslant 2$. This has allowed us to derive constraints which must be satisfied by quantum phases of matter on a lattice, such as superfluids and the more exotic Z_2 fractionalized phases which are topologically ordered. We have discussed ways in which these constraints may be useful in identifying fractionalized phases in numerical experiments.

A recurring theme has been the important distinction between even and odd deconfined Ising gauge theories, which correspond to states that are most naturally associated with integer and half integer filled systems, respectively. For exotic insulators, the even or odd character of the phase is completely determined by the filling in this manner. For exotic Fermi liquids and superfluids, a knowledge of the filling by itself is insufficient to determine the odd or even nature of the emergent Z_2 gauge field—precise violations of the Luttinger relation or its analogue for these systems provides a way to distinguish them from each other.

Within a simple Drude-like picture, one associates the size of the Fermi surface with the sign of the Hall conductivity—a Fermi surface corresponding to a few electrons would exhibit an electron-like response, while a Fermi surface of a nearly filled band would show a hole-like response. A Fermi surface which violates the conventional Luttinger theorem may thus be reflected in an anomalous sign of the Hall conductivity as depicted schematically in Fig. 5. A similar change in the sign of the Hall effect in a vortex liquid phase is expected for odd fractionalized superfluids relative to conventional superfluids, due to a shift of the Berry phase by π at a given density of bosons. Clearly, the sign of the Hall effect is not universal and in real systems is affected by band structure and interactions. Thus an anomalous sign of the Hall response is suggestive but is not a rigorous diagnostic. Experimental tools such as angle resolved photoemission spectroscopy which measure the Fermi surface can more directly detect violations of the conventional Luttinger theorem expected in odd Fermi liquids and thus serve to identify such systems.

ACKNOWLEDGMENTS

We thank L. Balents, M. P. A. Fisher, G. Misguich, M. Oshikawa, and T. Senthil for stimulating discussions. A.P. received support through Grant Nos. NSF DMR-9985255 and PHY99-07949 and the Sloan and Packard foundations. A.V. acknowledges support from a Pappalardo Fellowship.

APPENDIX A: DUALITY AND VORTONS IN $\mathcal{SF}^{*}$

Let us begin with the path integral for the partition function of chargons coupled to a Z_2 gauge field, $Z = \int \mathcal{D}\phi \Sigma_{\{n\}} \Sigma_{\{\sigma\}} \exp(-S)$, with $S = S_{gauge} + S_c$. The gauge field and chargon actions are given by

$$S_{gauge} = -\varepsilon K_s \prod_{\Box_s} \sigma_{ij} - \varepsilon K_\tau \prod_{\Box_\tau} \sigma_{ij},$$

$$S_c = -\varepsilon t_b \sum_{\langle ijra} \sigma_{ij} \cos(\phi_i - \phi_j) + \varepsilon U \sum_i (n_i - \overline{n})^2$$

$$+ i \sum_i n_i \left(\phi_i - \phi_{i+\tau} + \frac{\pi}{2}[1 - \sigma_{i,i+\tau}]\right). \tag{A1}$$

Here, $\langle ij \rangle$ denotes nearest neighbor sites in space, n, ϕ denote the chargon number and phase. The chargons hop with an amplitude t_b and have a local repulsion of strength U. $U\overline{n}$ plays the role of the chargon chemical potential. The Z_2 gauge fields are denoted by $\sigma_{ij} = \pm 1$ and $\Box_s$, $\Box_\tau$ denote elementary spatial/spatio-temporal plaquettes on the cubic space-time lattice with respective gauge field couplings K_s, K_τ. Finally, ε is the Trotter discretization along the imaginary time direction.

We can rewrite the chargon hopping term in the partition function as

$$\sum_{L_{ij}} e^{-\alpha \Sigma_{\langle ij \rangle} L_{ij}^2 + i \Sigma_{\langle ij \rangle} L_{ij}(\phi_i - \phi_j + \frac{\pi}{2}[1-\sigma_{ij}])}, \tag{A2}$$

where $L_{ij} = -L_{ji}$ is an integer-valued field. For large α, with $\alpha = \ln(2/\varepsilon t_b)$, this reduces to the original chargon hopping term. For general α, this modified form allows terms such as $\cos(2\phi_i - 2\phi_j)$ which correspond to chargon-pair hopping. We therefore do not need to keep an explicit pair-hopping term t_B unlike in our discussion in Sec. X.

Integrating out the phase field ϕ leads to a constraint

$$\sum_j L_{ij} + (n_i - n_{i+\tau}) = 0, \tag{A3}$$

which is juct the continuity equation, L_{ij} representing the chargon current on bond $(i \to j)$. Writing the number $n_i \equiv L_{i,i+\tau}$, we can recast this in the form

$$\sum_{\mu=0,1,2} [L_{i,i+x_\mu} + L_{i,i-x_\mu}] = 0, \tag{A4}$$

where $x_0 = \tau$, $x_1 = x$, $x_2 = y$. This sets the divergence of the 3 current to zero. Below, the sum over μ will be understood to run over 0,1,2 unless stated, we will also use the notation $L^i_\mu \equiv L_{i,i+x_\mu}$.

We go to dual vortex variables in 2+1 dimensions in the standard manner,[28] the only difference is in the presence of Z_2 gauge fields in the action but we do not dualize these. The constraint is solved by equating the conserved current to the curl of a dual vector such that its divergence is automatically zero. We decompose the chargon current into two parts, the current of pairs J (an even integer) and the current of unpaired particles I (=0,1). Note that I is only conserved modulo-2—two unpaired particles can combine to form a pair which is accommodated by increasing J by one unit, and I thus is the current of particles carrying only a Z_2 charge. The constraint is thus solved by choosing

$$J^i_\mu = 2\epsilon^{\mu\nu\lambda}\partial_\nu A^i_\lambda, \tag{A5}$$

$$I^i_\mu = (\epsilon^{\mu\nu\lambda}\partial_\nu a^i_\lambda)[\text{mod } 2], \tag{A6}$$

where A (an integar) and a (=0,1) are fields on links of the dual space-time lattice, and the right hand sides above are just the lattice curls on the dual lattice, taken around the original link $(i, i+x_\mu)$.

The chargon action then takes the form

$$S_c = \alpha \sum_{i,\mu=1,2} (J^i_\mu + I^i_\mu)^2 + \varepsilon U (J^i_0 + I^i_0 - \bar{n})^2 \tag{A7}$$

$$+ i\frac{\pi}{2}\sum_i I^i_0 (1 - \sigma_{i,i+\tau}) \tag{A8}$$

$$- \varepsilon g \sum (\epsilon^{0\nu\lambda}\partial_\nu a_\lambda)[\text{mod } 2], \tag{A9}$$

where we have now included a chemical potential g for the Z_2 charges whose density is $I^i_0 = \epsilon^{0\nu\lambda}\partial_\nu a_\lambda[\text{mod } 2]$.

We can convert the sum over A to an integral by softening the constraint by introducing terms $\varepsilon t_v \Sigma_{i,\mu}\cos(2\pi A^i_\mu)$ in the action (this can be formally accomplished by using Poisson resummation), which prefers A^i_μ to be an integer. Everywhere else in the action, only the transverse part of A plays a role (since only its lattice-curl appears). Extracting the longitudinal part of $2\pi A_{ij}$ as $\theta_i - \theta_j$, we identify the dual vorton creation operator $\exp(-i\theta_i)$. The vortons are seen to be minimally coupled to the transverse part of the A, which we denote $\mathcal{A}$, exactly as a charged particle coupled to a $U(1)$ gauge field. Thus, in the softened theory

$$2\pi A^i_\mu = (\theta_{i+x_\mu} - \theta_i) + 2\pi \mathcal{A}^i_\mu. \tag{A10}$$

Making this substitution, and absorbing a by shifting $\mathcal{A}^i_\mu \to \mathcal{A}^i_\mu - a^i_\mu/2$ find

$$S_c = \alpha \sum_{i,\mu=1,2} (\mathcal{J}^i_\mu)^2 + \varepsilon U \sum_i (\mathcal{J}^i_0 - \bar{n})^2 \tag{A11}$$

$$+ i\frac{\pi}{2}\sum_i I^i_0(1 - \sigma_{i,i+\tau}) - \varepsilon g \sum_i I^i_0 \tag{A12}$$

$$- t_v \sum_{i,\mu} \cos(\theta_i - \theta_{i,+x_\mu} - 2\pi\mathcal{A}^i_\mu - \pi a^i_\mu) \tag{A13}$$

with the total current $\mathcal{J}^i_\mu = J^i_\mu + I^i_\mu \equiv 2\epsilon_{\mu\nu\tau}\partial_\nu \mathcal{A}^i_\lambda$. This is the result used in Sec. X C.

[1] M. Yamanaka, M. Oshikawa, and I. Affleck, Phys. Rev. Lett. **79**, 1110 (1997).
[2] M. Oshikawa, Phys. Rev. Lett. **84**, 3370 (2000).
[3] J. M. Luttinger, Phys. Rev. **119**, 1153 (1960).
[4] G. Misguich, C. Lhuillier, B. Bernu, and C. Waldtmann, Phys. Rev. B **60**, 1064 (1999); G. Misguich, C. Lhuillier, M. Mambrini, and P. Sindzingre, Eur. Phys. J. B **26**, 167 (2002).
[5] D. N. Sheng and L. Balents (unpublished).
[6] T. Senthil, S. Sachdev, and M. Vojta, Phys. Rev. Lett. **90**, 216403 (2003).
[7] F. D.M. Haldane and Y. Wu, Phys. Rev. Lett. **55**, 2887 (1985).
[8] P. Ao and D. J. Thouless, Phys. Rev. Lett. **70**, 2158 (1993).
[9] T. Senthil and M. P.A. Fisher, Phys. Rev. B **62**, 7850 (2000).
[10] A. Paramekanti, N. Trivedi, and M. Randeria, Phys. Rev. B **57**, 11639 (1998).
[11] R. Moessner, S. L. Sondhi, and E. Fradkin, Phys. Rev. B **65**, 024504 (2002).
[12] R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981).
[13] C. Wexler, Phys. Rev. Lett. **79**, 1321 (1997).
[14] M. Oshikawa, Phys. Rev. Lett. **90**, 236401 (2003); **91**, 109901 (Errata) (2003).
[15] M. B. Hastings, Phys. Rev. B **69**, 104431 (2004).
[16] See D. J. Thouless, P. Ao, and Q. Niu, Phys. Rev. Lett. **76**, 3758 (1996); G. E. Volovik, *ibid.* **77**, 4687 (1996), E. B. Sonin, *ibid.* **81**, 4276 (1998), C. Wexler, D. J. Thouless, P. Ao and Q. Niu, *ibid.* **81**, 4277 (1998).
[17] Note that we *define* an insulator as $\bar{I}=0$ which is a stronger condition than a vanishing Drude weight used by Oshikawa (see Ref. 14). In particular, a system with a finite zero frequency conductivity would be an insulator by his definition but not by the more stringent condition used here.
[18] M. Oshikawa, Phys. Rev. Lett. **84**, 1535 (2000).

[19] D. H. Lee and R. Shankar, Phys. Rev. Lett. **65**, 1490 (1990).

[20] The total number of such quasidegenerate states depends on the pattern of symmetry breaking in the thermodynamic limit. However, all these states may not be accessible from the ground state by flux threading; the momentum counting argument provides a constraint on which states can be accessed. For instance, in $D=2$, only states which differ from the ground state by crystal momentum $2\pi(p/q)L_y$ can be accessed when threading flux along the y direction.

[21] However, they can still cross an even number of times at other intermediate flux values which, from time-reversal symmetry, are symmetric about $\Phi=\pi$.

[22] R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001); L. Balents, M. P.A. Fisher, and S. M. Girvin, Phys. Rev. B **65**, 224412 (2002); G. Misguich, D. Serban, and V. Pasquier, Phys. Rev. Lett. **89**, 137202 (2002); O. Motrunich and T. Senthil, *ibid.* **89**, 277004 (2002). A. Kitaev, Ann. Phys. (San Diego) **303**, 2 (2003); X.-G. Wen, Phys. Rev. Lett. **90**, 016803 (2003).

[23] This is to be viewed as choice of a reference state which we can then project into the subspace of physical states which satisfy the constraint (21).

[24] N. E. Bonesteel, Phys. Rev. B **40**, 8954 (1989).

[25] P. W. Anderson, Science **235**, 1196 (1987).

[26] D. Ivanov and T. Senthil, Phys. Rev. B **66**, 115111 (2002); A. Paramekanti, M. Randeria, and N. Trivedi, cond-mat/0303360, cond-mat/0405353.

[27] We have also checked the vison momentum to be expected on a triangular lattice on even×even and odd×even geometries. On even×even lattices, it is still zero, while on odd×even lattices it carries a momentum which is half of the reciprocal lattice vector.

[28] M. P.A. Fisher and D. H. Lee, Phys. Rev. B **39**, 2756 (1989).

[29] A different way to view $\mathcal{SF}^*$ is to consider a weakly coupled bilayer system consisting of two parts: (i) a fractionalized insulating layer $\mathcal{I}^*$ which supports gapped visons and chargons and (ii) a condensate of charge-Q bosons which is a conventional superfluid ($\mathcal{SF}$) supporting hc/Q vortices. In the absence of any coupling between the layers, this bilayer system clearly supports all these separate excitations. Imagine turning on a weak coupling which allows transfer of charge between the two layers, via processes which involve two chargons in the $\mathcal{I}^*$-layer pairing and becoming part of the condensate in the superfluid layer. The chargon is then no longer a well-defined excitation in the system since its electric charge can be screened by the condensate. The physical excitation is in fact an electrically neutral remnant of a single chargon, which carries only a Z_2 Ising gauge charge, the ison. The vison and the hc/Q vortex continue to be good excitations.