Introduction to the Renormalization Group Flow

Daniel Friedan

Department of Physics and Astronomy
Rutgers, The State University of New Jersey

The Natural Science Institute
University of Iceland

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Example: the 2d nonlinear model

Given a manifold M, there is a family of 2d quantum field theories (QFTs) parametrized by the Riemannian metrics $\frac{1}{\alpha'} g_{ij}$ on M. (I'll often write g for $\frac{1}{\alpha'} g_{ij}$.)

The QFT is:

for every $2d$ surface Σ with metric $\gamma_{\mu\nu}$, a measure $d\rho$ on the space of maps $\phi : \Sigma \to M$

$$d\rho(\gamma, g; \phi) = \mathcal{D}\phi \; e^{-S(\phi)}$$

where

$$\begin{aligned} S(\phi) &= \|d\phi\|^2 = \int_\Sigma \mathrm{dvol}_\gamma(x) \, \frac{1}{\alpha'} g_{ij}(\phi(x)) \, \partial^\mu \phi^i(x) \, \partial_\mu \phi^j(x) \\ \mathcal{D}\phi &= \prod_{x \in \Sigma} \mathrm{dvol}_g(\phi(x)) \end{aligned}$$

This is actually well-defined as a formal power series in α' (a "perturbative" quantum field theory).

Notice that the action $S(\phi)$ does not depend on the scale of $\gamma_{\mu\nu}$.

We usually just take $\Sigma = \mathbb{R}^2$ with euclidean metric $\gamma_{\mu\nu}$. I'll try to explain why.

The measure is encoded in its integrals (measurements), e.g.

$$\int \mathcal{D}\phi \; e^{-S(\phi)} \; F_1(\phi(x_1)) \cdots F_n(\phi(x_n) \qquad F_k \in C^\infty(M) \quad x_k \in \Sigma \, .$$

The renormalization group

The renormalization group (RG) is a flow on the space of QFTs, $t \to QFT_t$, generated by a vector field β, such that:

QFT_t in 2d metric $e^{-2t}\gamma_{\mu\nu}$ predicts the same measurements as QFT_0 in $\gamma_{\mu\nu}$.

For the nonlinear model

$$\frac{d}{dt}\left(\frac{1}{\alpha'}g_{ij}\right)_t = -\beta_{ij}(g_t) \qquad \beta_{ij}(g) = R_{ij} + O(\alpha')$$

The QFTs are covariant under the flow:

$$\frac{d}{dt}\, d\rho(e^{-2t}\gamma, g_t) = 0$$

On $\mathbb{R}^2$ with $\gamma_{\mu\nu} = \Lambda^2\delta_{\mu\nu}$, we can write this RG group equation

$$\frac{d}{dt}\, d\rho(e^{-2t}\Lambda^2, g_t) = 0$$

That is, if we scale the unit of distance larger $\Lambda^{-1} \to e^t\Lambda^{-1}$ and at the same time let the parameters of the qft flow under the RG, nothing changes.

Another way to say this: if we flow under the RG, everything in the QFT becomes smaller. Then we make the unit of distance larger and everything looks the same as before.

The Ricci flow from the RG flow

The RG flow

$$\frac{d}{dt}\left(\frac{1}{\alpha'}g_{ij}\right)_t = -R_{ij} + O(\alpha')$$

in the limit $\alpha' \to 0$ does not become the Ricci flow.

We have to re-scale the RG "time" to $a = \alpha' t$ to get

$$\frac{d}{da}\, g_{ij} = -R_{ij} + O(\alpha')$$

A solution of this re-scaled RG equation is of the form

$$\tilde{g}_{ij}(a) + \sum_{n=1}^{\infty} \alpha'^n \Delta\tilde{g}^n_{ij}(a)$$

where $\tilde{g}_{ij}(a)$ is a Ricci flow and $\Delta\tilde{g}^n_{ij}(0) = 0$. Then the perturbative solution of the RG equation would be

$$\frac{1}{\alpha'}\tilde{g}_{ij}(0) + t\tilde{g}'_{ij}(0) + \alpha'\left[\frac{1}{2}t^2(\tilde{g}''_{ij}(0) + t(\Delta\tilde{g}^1_{ij})'(0)\right] + \cdots$$

At each order in α' the RG flow is polynomial in t.

Another difference: we are quite sure that the RG flow is eternal for $M = S^n$ with round metric, while the Ricci flow ends in finite time.

Lattice regularization

"Regularize" means to replace the formal integral over maps by an approximation that makes sense.

For example, approximate a square 2-torus of side L with a lattice $\Sigma = \mathbb{Z}_N \times \mathbb{Z}_N$, The distance between neighboring points is L/N.

The space of maps $\Sigma \to M$ is just M^{N^2}.

Approximate the energy functional, for example by a sum over nearest neighbors

$$S_{lattice}(\phi) = \sum_{(x,x')} \mathrm{dist}^2_{\frac{1}{\alpha'} g_{ij}}(\phi(x), \phi(x'))$$

The functional volume element is just the metric volume element on M^{N^2}

$$\mathcal{D}\phi = \prod_{x \in \Sigma} \mathrm{dvol}_g(\phi(x))$$

The question is: can we take the continuum limit $N \to \infty$?

Also: do all regularizations give the same continuum limit?

Note that the integral depends only on N, not on L (2d scale invariance).

Why *quantum* field theory?

An integral over paths in $\mathcal{N}$ is equivalent to a quantum mechanics on $L_2(\mathcal{N})$

$$\int_{\substack{\text{paths } \Phi(\tau) \\ \Phi(0)=\Phi_0 \\ \Phi(T)=\Phi_1}} \mathcal{D}\Phi \; e^{-S(\Phi)} = \langle \Phi_1 | e^{-TH} | \Phi_0 \rangle = \langle \Phi_1 | e^{-itH} | \Phi_0 \rangle$$

Analytically continue to $T = it$ to get the kernel (matrix-elements) of the quantum mechanical time evolution operator e^{-itH}, where H is the hamiltonian.

For the 2d nonlinear model, let the surface $\Sigma = \mathbb{R} \times S$ where S is 1-dimensional.

The maps $\phi : \Sigma \to M$ are the paths in $\mathcal{N} = \text{Maps}(S \to M)$.

The QFT is a quantum mechanics on $L_2(\text{Maps}(S \to M))$.

It's called a quantum *field* theory because its operators such as $F(\phi(x,\tau))$ depend on the spatial position $x \in S$, as well as the time τ, and commute for different x.

In string theory, take $S = S^1$ to get a quantum mechanics for a closed string moving in a spacetime M with spacetime metric $\frac{1}{\alpha'} g_{ij}$.

Perturbation theory (formal)

For $\alpha' \sim 0$, the measure is dominated by the constant maps $\phi(x) = \phi_0 \in M$.

Around each $\phi_0 \in M$, choose coordinates ϕ^i in $T_{\phi_0}M$:

$$\phi^i(\phi_0) = \phi_0^i \qquad \phi^i(\phi(x)) = \phi_0^i + \pi^i(x)$$

The integral is now over the constants $\phi_0 \in M$ and the fluctuations $\pi^i(x)$ (modulo the constant $\pi^i(x)$).

$$S(\phi) = S(\phi_0;\pi) = \int_\Sigma \mathrm{dvol}_\gamma(x)\, \frac{1}{\alpha'} g_{ij}(\phi_0 + \pi(x))\, \partial^\mu \pi^i(x)\, \partial_\mu \pi^j(x)$$

$$\int \mathcal{D}\phi\, e^{-S(\phi)}\, (\cdots) = \int_M \mathrm{dvol}_g(\phi_0) \int_V \mathcal{D}\pi\, e^{-S(\phi_0;\pi)}\, (\cdots)$$

V is the vector space of maps $\pi : \Sigma \to T_{\phi_0}M$ (modulo the constant maps).

The integral over V is very close to a gaussian integral:

$$\tilde\pi^i(x) = (\alpha')^{-1/2}\pi^i(x)$$

$$S(\phi_0,\pi) = \int_\Sigma \mathrm{dvol}_\gamma(x)\, g_{ij}(\phi_0)\, \partial^\mu \tilde\pi^i(x)\, \partial_\mu \tilde\pi^j(x) \;+\; O((\alpha')^{1/2})$$

The Feynman diagrams organize the perturbative calculation of nearly gaussian integrals over vector spaces.

Regularize

Approximate the integration space V by the subspace V_{t_0} on which

$$-\nabla^\mu \partial_\mu = \Delta < e^{-2t_0} \qquad t_0 \ll 0 \qquad e^{t_0} \ll 1$$

We would like to take the limit $t_0 \to -\infty$.

Label the metric g_{t_0}. The regularized (cutoff) measure is

$$\int d\rho_{t_0}(\gamma, g_{t_0}; \phi)\,(\cdots) = \int\limits_M \mathrm{dvol}_{g_{t_0}}(\phi_0) \int\limits_{V_{t_0}} \mathcal{D}\pi_0 \; e^{-S(g_{t_0}, \phi_0; \pi_0)}\,(\cdots)$$

The integration space V_{t_0} is finite dimensional if Σ is compact.

If $\Sigma = \mathbb{R}^2$ then V_{t_0} is still infinite dimensional. This is the *infrared* problem. We won't actually have to face it.

Renormalize

Take $\delta > 0$ very small. Let $V_{t_0,t_0+\delta}$ be the subspace of short-distance fluctuations

$$e^{-2(t_0+\delta)} < \Delta < e^{-2t_0}$$

The integration space decomposes: $\qquad V_{t_0} = V_{t_0+\delta} \oplus V_{t_0,t_0+\delta}$

$$\pi_0(x) = \pi(x) + \pi'(x) \qquad \pi_0 \in V_{t_0} \quad \pi \in V_{t_0+\delta} \quad \pi' \in V_{t_0,t_0+\delta}$$

We can integrate out the the short-distance fluctuations as long as the functions being integrated depend only on the $\pi \in V_{t_0+\delta}$. (We only take measurements at 2d distances larger than e^{t_0}.)

$$\begin{aligned}
\int\limits_{V_{t_0}} \mathcal{D}\pi_0 \; e^{-S(g_{t_0},\phi_0;\pi_0)} \, (\cdots) \quad &= \int\limits_{V_{t_0+\delta}} \mathcal{D}\pi \int\limits_{V_{t_0,t_0+\delta}} \mathcal{D}\pi' \; e^{-S(g_{t_0},\phi_0;\pi+\pi')} \, (\cdots) \\
&= \int\limits_{V_{t_0+\delta}} \mathcal{D}\pi \; e^{-S'(g_{t_0},\phi_0;\pi)} \, (\cdots)
\end{aligned}$$

where

$$e^{-S'(g_{t_0},\phi_0;\pi)} = \int\limits_{V_{t_0,t_0+\delta}} \mathcal{D}\pi' \; e^{-S(g_{t_0},\phi_0;\pi+\pi')}$$

Next, we argue that the new action takes the same form as the old

$$S'(g_{t_0}, \phi_0; \pi) = S(g_{t_0+\delta}, \phi_0; \pi) + O(e^{t_0})$$

for some slightly changed metric on M

$$g_{t_0+\delta} = g_{t_0} - \delta \cdot \beta(g_{t_0})$$

$g_{t_0+\delta}$ is calculated in the form of a Taylor series around $\phi_0 \in M$. We do this for each ϕ_0. Then we show that the resulting Taylor series all come from a single metric $g_{t_0+\delta}$ on M.

So we have

$$\int_M d\phi_0 \int_{V_{t_0}} \mathcal{D}\pi_0 \; e^{-S(g_{t_0}, \phi_0; \pi_0)} \, (\cdots) = \int_M d\phi_0 \int_{V_{t_0+\delta}} \mathcal{D}\pi \; e^{-S(g_{t_0+\delta}, \phi_0; \pi)} \, (\cdots)$$

which we write

$$d\rho_{t_0}(\gamma, g_{t_0}; \phi) = d\rho_{t_0+\delta}(\gamma, g_{t_0+\delta}; \phi)$$

Some points about integrating out the short-distance fluctuations:

(1) $\Delta \approx e^{-2t_0} \gg 1$ so Σ might as well be euclidean $\mathbb{R}^2$

(2) The integrating out can be done effectively, order by order in α', as a sum of Feynman diagrams, each a bounded integral of a bounded function.

(3) $\beta(g_{t_0})$ depends only on g_{t_0}, not on Σ or $\gamma_{\mu\nu}$ (since $e^{2t_0}\Delta \approx 1$).

(4) The new metric $g_{t_0+\delta}$ is constructed covariantly wrt *Diff*(M).

(5) $\beta(g)$, does not depend on any of the arbitrary choices, such as coordinate systems or method of regularization, up to equivalence under *Diff*(M).

Changing these can only change β by a vertical vector field.

Now iterate this infinitesimal process to obtain, for $t > t_0$

$$d\rho_{t_0}(\gamma, g_{t_0}; \phi) = d\rho_t(\gamma, g_t; \phi)$$

where

$$\frac{dg_t}{dt} = -\beta(g_t)\,.$$

Now suppose we can integrate the flow *backwards* in t_0. In perturbation theory, we can in fact integrate the flow backwards, because g_{t_0} is polynomial in t_0 at each order in α'.

Then a continuum limit exists

$$\lim_{t_0 \to -\infty} d\rho_{t_0}(\gamma, g_{t_0}; \phi) = d\rho_t(\gamma, g_t; \phi)$$

parametrized by t, g_t. We might as well parametrize the continuum limit by $g_0 = g$

The choice of t was arbitrary, so this continuum QFT is defined on all maps ϕ.

The cutoff is $e^{2t_0}\Delta < 1$ and the action is scale invariant, so

$$d\rho_{t_0}(\gamma, g_{t_0}; \phi) = d\rho_0(e^{-2t_0}\gamma, g_{t_0}; \phi)$$

so we can write the continuum limit as

$$\begin{aligned} d\rho(\gamma, g; \phi) &= \lim_{t_0 \to -\infty} d\rho_0(e^{-2t_0}\gamma, g_{t_0}; \phi) \\ &= \lim_{t_0 \to -\infty} d\rho_0(e^{-2t_0-2t}\gamma, g_{t_0+t}; \phi) \\ &= d\rho(e^{-2t}\gamma, g_t; \phi) \end{aligned}$$

which is the RG covariance of the QFT.

Construction of non-perturbative QFTs

We want to construct honest QFTs, not just perturbative ones. For this, we need to run the RG flow in reverse. Moreover, we need some control over the behavior as $t \to -\infty$.

The only nonlinear models that are easily controlled are at $\alpha' \approx 0$.

There are fixed points $\frac{1}{\alpha'} g_{ij}$ at $\alpha' = 0$ for

$$R_{ij} - \lambda g_{ij} = (\mathcal{L}_v g)_{ij} = \nabla_i v_j + \nabla_j v_i$$

for some vector field v on M. The rhs expresses the fact that β_{ij} is defined only up to infinitesimal diffeomorphisms of M, that the RG flow actually acts on the space of metrics modulo *Diff* (M).

For $\lambda > 0$, the fixed point at $\alpha' = 0$ is repulsive in the α direction, so the RG flow can be run backwards forever.

For $\lambda = 0$, the same is true because of the $O(\alpha')$ term in β_{ij}.

For $\lambda < 0$, the fixed point is attractive in the $\alpha' = 0$ direction. These describe limits of the RG flow as $t \to +\infty$.

Ancient solutions of the Ricci flow might give new QFTs, if there is enough control of the limit $t \to -\infty$. It would be necessary to show stability against the terms at higher order in α'.

Gradient formulas

Let's switch to a more abstract notation. Let λ^I be coordinates on the space of QFTs. For the nonlinear model, the λ^I are coordinates on the space of metrics g. The RG flow is

$$\frac{d\lambda^I}{dt} = -\beta^I(\lambda)\,.$$

A gradient formula would be

$$\frac{\partial F}{\partial \lambda^I} = G_{IJ}(\lambda)\beta^J(\lambda)$$

for some function F and some riemannian metric G_{IJ} on the space of QFTs.

The fixed points $\beta = 0$ would then be the critical points of F.

F would decrease under the RG flow:

$$\frac{dF(\lambda)}{dt} = \frac{\partial F}{\partial \lambda^I}\frac{d\lambda^I}{dt} = -G_{IJ}\beta^I\beta^J \leq 0\,.$$

String theory

The fixed point equation $\beta_{ij} = 0$ looked like Einstein's equation in general relativity, the classical theory of gravity. Einstein's equation comes from an action principle: its solutions are the critical points of the Einstein-Hilbert action. So I spent some early efforts on looking for a gradient formula for β_{ij}, with little success.

In string theory, a 2d QFT on a surface $\Sigma = \mathbb{R} \times S^1$ describes the quantum mechanics of a string.

The scale-invariance condition $\beta = 0$ is a technical consistency condition on the string quantum mechanics.

Soon after the renormalizability of the nonlinear model was demonstrated, it was realized that nonlinear models described the quantum mechanics of a string moving in a space-time manifold M, in a background gravitational metric $\frac{1}{\alpha'} g_{ij}$.

For string theory, a term was added to the 2d action $S(\phi)$ proportional to the 2d scalar curvature

$$\int_\Sigma \mathrm{dvol}_\gamma(x)\; R_\gamma\; D(\phi(x))\,.$$

The *dilaton* function D on M provided an additional set of parameters: a larger family of QFTs.

We calculated $\beta^I(\lambda)$ for this expanded set of parameters, to leading order in α', and found an action $F(\lambda)$ such that the critical points of F were the solutions of $\beta^I(\lambda) = 0$.

The c-theorem

There are axiomatic treatments of QFT, on $\mathbb{R}^2$ in particular, so we can talk of the space of QFTs in the abstract. We have many examples, some exactly soluble, and some general knowledge of this space.

The c-theorem says that, for euclidean metrics $\gamma_{\mu\nu}$ on $\mathbb{R}^2$, there is a function $c(\gamma, \lambda)$ such that

$$\begin{aligned} c(\gamma, \lambda) &\geq 0 \\ -\frac{d}{dt} c(e^{-2t}\gamma, \lambda) &\leq 0 \end{aligned}$$

If $c(\gamma, \lambda)$ is covariant under the RG, then the second inequality is equivalent to

$$\frac{d}{dt} c(\gamma, \lambda_t) \leq 0$$

There is an argument that c should be RG-covariant, but not a completely general or rigorous argument. To get RG-covariance in general, in enough generality to apply to the nonlinear model, it is necessary to add parameters λ^I playing in the abstract the same role as the dilaton function $D(\phi)$. This might spoil the condition $c \geq 0$.

The c-theorem looks like it should have come from a gradient formula, but in fact it did not.

It is still open whether there is a general gradient formula for the RG flow on the space of 2d QFTs.

Boundary QFT

QFT can be done on surfaces Σ with boundary. The basic cases are $\Sigma = \mathbb{R} \times [0, \infty)$ and $\Sigma = S^1 \times [0, \infty)$ In string theory, $\Sigma = \mathbb{R} \times I$ is used for the open string.

Additional parameters λ^a describe the boundary behavior. They also flow under the RG,

$$\frac{d\lambda^a}{dt} = -\beta^a(\lambda)$$

What has been studied mostly are the situations where the bulk QFT is at a fixed point, $\beta^I = 0$, so only the boundary parameters λ^a can flow.

The nonlinear model provides examples: mean curvature flow and Donaldson flow (most likely neither in complete generality).

Anatoly Konechny and I proved a general gradient formula on the space of boundary QFTs (with bulk $\beta^I = 0$)

$$\frac{\partial s(\lambda)}{\partial \lambda^a} = G_{ab}(\lambda)\beta^b(\lambda)$$

where $s(\lambda)$ is literally the quantum mechanical entropy in the boundary at a given fixed temperature and G_{ab} is a riemannian metric on the space of boundary QFTs. So

$$\frac{ds}{dt} \leq 0$$

and is stationary exactly at the fixed points. We have not been able to put any lower bound on s.