Finale Exam

PHYS 677,

Fall 2018

1. Consider a 1D random walk with Gaussian step lengths:

[10 points]

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

↑ length of a single displacement

Find $p_n(R)$, the probability density for the position of the random walker after the nth step : $R = x_1 + x_2 + \ldots + x_n$

Hint: use the method of the characteristic function.

Note: Work out the exact expression for $p_n(R)$ valid at all n, not just in the $n \to \infty$ limit.

(2.) Use transfer matrix techniques

[20 points] in a 1D Ising model with $H = 0$:

$$\mathcal{H} = -J \sum_{i=0}^{N-1} S_i S_{i+1}$$

and periodic boundary conditions, find the two-spin correlation function,

$$\Gamma_R = \langle S_0 S_R \rangle - \langle S_0 \rangle \langle S_R \rangle$$

↑ spin position

and the correlation length ξ:

$$\xi^{-1} = \lim_{R \to \infty} \left\{ -\frac{1}{R} \log |\Gamma_R| \right\}.$$

in the thermodynamic limit $(N \to \infty)$.

Use these results to find T_c in this system.

(3.) Recall Langevin's desription of the dynamics of a Brownian particle:
[20 points]

external force

$$\underset{\substack{\uparrow \\ \text{particle mass}}}{m}\frac{d\vec{v}}{dt} = -\underset{\substack{\uparrow \\ \text{friction coeff.}}}{\gamma}\vec{v} + \vec{F} + \underset{\substack{\uparrow \\ \text{stochastic force}}}{\vec{\eta}(t)}$$

It can be shown that ($m=1$ for simplicity)

$$\langle \eta_i(t) \rangle = 0,$$

$$\langle \eta_i(t)\,\eta_j(t') \rangle = \Gamma\,\delta_{ij}\,\delta(t-t'),$$

where $\Gamma = 2k_B T\gamma$.

In the so-called overdamped limit, inertial effects captured by the $m\frac{d\vec{v}}{dt}$ term may be disregarded,

leading to: $\frac{dx}{dt} = \frac{F}{\gamma} + \eta(t)$,
(restrict to <u>1D</u> for simplicity)

where $F = -\frac{\partial V}{\partial x}$ and

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\,\eta(t') \rangle = \frac{2k_B T}{\gamma}\,\delta(t-t').$$

In other words,

$$\frac{dx}{dt} = -\frac{1}{\gamma}\frac{\partial V}{\partial x} + \sqrt{2D}\,\tilde{\eta}(t),$$

where $D = \frac{k_B T}{\gamma}$ is the diffusion constant and $\langle \tilde{\eta}(t) \rangle = 0$, $\langle \tilde{\eta}(t)\,\tilde{\eta}(t') \rangle = \delta(t-t')$.

Write down the equivalent Fokker-
- Planck equation for $P(x,t)$, the prob.
of the particle's position x at time t.

Find the steady-state distribution
$P_s(x)$ and time-dependent moments
$\langle x(t) \rangle$ and $\langle x^2(t) \rangle$ (assume that
the particle is at x_0 at $t=0$),
for a quadratic potential

$$V(x) = \frac{\alpha x^2}{2}.$$