Energy in Bremsstrahlung, Note on p. 179

From homework #2 we learned that the energy density is the $(0,0)$ component of the Noether current associated with translations. Generally, associated with $\delta x^\mu = g^{\mu\nu}$ and $\delta A_\mu = 0$, we have

$$T^{\nu\mu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\rho}\left(\partial_\sigma A_\rho \delta x^\sigma - \delta A_\rho\right) - \mathcal{L}\delta x^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\rho}\partial_\sigma A_\rho g^{\sigma\nu} - \mathcal{L}\delta g^{\mu\nu}.$$

From homework #1 we have

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

so

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\rho} = F^{\rho\mu}.$$

Thus

$$T^{\nu\mu} = F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma},$$

and

$$\mathcal{H} = T^{00} = \frac{1}{4}\left(F^{\rho\mu}F_{\rho\mu} - 2F^{0\mu}F_{0\mu}\right) = \frac{1}{2}\left(F^{0j}F_{0j} - F^{ij}F_{ij}\right) = \frac{1}{2}\left(E^2 + B^2\right).$$

Note that we can write $\mathcal{H}$ in the strange fashion

$$\mathcal{H} = -\frac{1}{4}\sum_\rho F^{\rho\mu}F^\rho{}_\mu,$$

where I have explicitly given the $\sum_\rho$, because we don't have a summation convention on two upper indices, but the $\sum_\mu$ is left implicit.

As discussed on pages 178-179 in Peskin and Schroeder, the radiation field in the impulse approximation for electron scattering is given by the residues of the poles at $k^0 = \pm|\vec{k}|$ in

$$A^\mu(x) = \int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\int\frac{d^3k}{(2\pi)^3}e^{-ik\cdot x}\frac{-ie}{k^2}\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)$$

The residues of $1/k^2 = 1/[(k^0 - |\vec{k}|)(k^0 + |\vec{k}|)$ are $\pm(1/2|\vec{k}|)$ at $k^0 = \pm|\vec{k}|$ respectively, so

$$\begin{aligned} A^\mu_{\rm rad}(x) &= \int\frac{d^3k}{(2\pi)^3}\frac{-e}{2|\vec{k}|}\Bigg\{\left(\frac{p'^\mu}{k^0E' - \vec{k}\cdot\vec{p}'} - \frac{p^\mu}{k^0E - \vec{k}\cdot\vec{p}}\right)e^{-i(k^0t - \vec{k}\cdot\vec{x})} \\ &\qquad - \left(\frac{p'^\mu}{-k^0E' - \vec{k}\cdot\vec{p}'} - \frac{p^\mu}{-k^0E - \vec{k}\cdot\vec{p}}\right)e^{-i(-k^0t - \vec{k}\cdot\vec{x})}\Bigg\}\Bigg|_{k^0=+|\vec{k}|} \end{aligned}$$

Reversing the sign of the integration variable $\vec{k}$ in the second term, this may be written as

$$\begin{aligned} A^\mu_{\rm rad}(x) &= \int\frac{d^3k}{(2\pi)^3}\frac{-e}{2|\vec{k}|}\Bigg\{\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)e^{-ik\cdot x} \\ &\qquad + \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)e^{ik\cdot x}\Bigg\}\Bigg|_{k^0=+|\vec{k}|} \\ &= \int\frac{d^3k}{(2\pi)^3}\frac{-e}{2|\vec{k}|}\Bigg\{\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)e^{-ik\cdot x} + c.c.\Bigg\}\Bigg|_{k^0=+|\vec{k}|} \end{aligned}$$

in agreement with 6.6 of Peskin and Schroeder. Thus we have

$$A^\mu_{\rm rad}(x) = \int\frac{d^3k}{2(2\pi)^3}\mathcal{A}^\mu(\vec{k})\left(e^{-ik\cdot x} + e^{ik\cdot x}\right), \text{ with } \mathcal{A}^\mu(\vec{k}) = \frac{-e}{|\vec{k}|}\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right).$$

Note $k_\mu\mathcal{A}^\mu(\vec{k}) = 0$ for $k^0 = |\vec{k}|$. If we define $\bar{k}^\mu := (-|\vec{k}|, \vec{k})$ (so $\bar{k}_\mu = -\bar{k}^\mu$), we also have $\bar{k}_\mu\mathcal{A}^\mu(-\vec{k}) = 0$. This will prove useful in evaluating the energy in the radiation field.

From the expression for $A^\mu_{\rm rad}(x)$ we have

$$F^{\mu\nu}_{\rm rad}(x) = -i\int\frac{d^3k}{2(2\pi)^3}\left(k^\mu\mathcal{A}^\nu(\vec{k}) - k^\nu\mathcal{A}^\mu(\vec{k})\right)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right)\Big|_{k^0=|\vec{k}|}.$$

so

$$\begin{aligned} H &= -\frac{1}{4}\int d^3x\sum_\rho F^{\rho\mu}_{\rm rad}F_{\rm rad}{}^\rho{}_\mu \\ &= \frac{1}{4}\int d^3x\int\frac{d^3k}{2(2\pi)^3}\int\frac{d^3k'}{2(2\pi)^3} \\ &\qquad \sum_\rho\left(k^\rho\mathcal{A}^\mu(\vec{k}) - k^\mu\mathcal{A}^\rho(\vec{k})\right)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) \\ &\qquad \times\left(k'^\rho\mathcal{A}_\mu(\vec{k}') - k'^\mu\mathcal{A}_\rho(\vec{k}')\right)\left(e^{-ik'\cdot x} - e^{ik'\cdot x}\right)\Big|_{\substack{k^0=|\vec{k}| \\ k'^0=|\vec{k}'|}} \end{aligned}$$

We can use $\int d^3x d^3k' e^{i(\vec{k}\pm\vec{k}')\cdot\vec{x}} = (2\pi)^3$ with $\mp\vec{k}$ substituted for $\vec{k}'$ in the rest of the expression. As all k^0 and k'^0 are positive, it would be better to say $k' \to -\bar{k}$ in the first case.
Thus we have

$$\begin{aligned} H &= \frac{1}{4}\int \frac{d^3k}{4(2\pi)^3}\sum_\rho \Big\{ -2\left(k^\rho \mathcal{A}^\mu(\vec{k}) - k^\mu \mathcal{A}^\rho(\vec{k})\right)\left(k^\rho \mathcal{A}_\mu(\vec{k}) - k^\mu \mathcal{A}_\rho(\vec{k})\right) \\ &\quad + \left(k^\rho \mathcal{A}^\mu(\vec{k}) - k^\mu \mathcal{A}^\rho(\vec{k})\right)\left(\bar{k}^\rho \mathcal{A}_\mu(-\vec{k}) - \bar{k}^\mu \mathcal{A}_\rho(-\vec{k})\right) \\ &\qquad \times \left(e^{2i|\vec{k}|t} + e^{-2i|\vec{k}|t}\right)\Big\}\Big|_{k^0=|\vec{k}|}. \end{aligned}$$

where $\bar{k} = (-k^0, \vec{k})$. Note $\sum_\rho k^\rho V^\rho = -\bar{k}\cdot V$, and $\sum_\rho k^\rho k^\rho = 2|\vec{k}|^2$, so

$$\begin{aligned} H &= \frac{1}{16}\int \frac{d^3k}{(2\pi)^3}\Big\{ -8|\vec{k}|^2 \mathcal{A}^2(\vec{k}) + 4k\cdot\mathcal{A}(\vec{k})\bar{k}\cdot\mathcal{A}(\vec{k}) \\ &\quad + \left(-2k^2\mathcal{A}(\vec{k})\cdot\mathcal{A}(-\vec{k}) + 2k\cdot\mathcal{A}(-\vec{k})k\cdot\mathcal{A}(\vec{k})\right)\left(e^{2i|\vec{k}|t} + e^{-2i|\vec{k}|t}\right) \\ &= -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|\vec{k}|^2\mathcal{A}^2(\vec{k}) \end{aligned}$$

where we have used $k^2 = 0$ and $k\cdot\mathcal{A}(\vec{k}) = 0$. Thus we have

$$\begin{aligned} H &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|\vec{k}|^2\left(\frac{-e}{|\vec{k}|}\right)^2\left(\frac{2p\cdot p'}{k\cdot p'\,k\cdot p} - \frac{m^2}{(k\cdot p')^2} - \frac{m^2}{(k\cdot p)^2}\right) \\ &= \frac{e^2}{2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{2p\cdot p'}{k\cdot p'\,k\cdot p} - \frac{m^2}{(k\cdot p')^2} - \frac{m^2}{(k\cdot p)^2}\right) \end{aligned}$$

in agreement with 6.13.

On getting 6.17:
For extreme relativistic $\vec{v}$ and $\vec{v}'$, the integral over the angles of $\hat{k}$ is dominated by the region around each of the two $\vec{v}$'s. With θ the angle between $\hat{k}$ and $\vec{v}$, we will integrate from $\theta = 0$, but we don't want to get too close the region near $\vec{v}'$, as we wish to approximate $1-\hat{k}\cdot\vec{v}' \approx 1-\vec{v}\cdot\vec{v}'$. So we will go halfway, up to $\cos\theta = 1 - \frac{1}{2}(1-\vec{v}\cdot\vec{v}')$ and write $I(\vec{v},\vec{v}') \approx \frac{1}{|v|}\ln(1-vu)\Big|^1_{\frac{1+\vec{v}\cdot\vec{v}'}{2}} = \frac{1}{|v|}\left(\ln\frac{1}{2} + \ln\frac{1-\vec{v}\cdot\vec{v}'}{1-|v|}\right)$, plus the similar contribution from $\vec{v}\leftrightarrow\vec{v}'$. This justifies 6.17 in the ultrarelativistic limit.