An extra note on Noether’s Theorem

I was asked to clarify the connection of conserved charges emerging as a consequence of a symmetry and the generator of those symmetry transformations.

Let us consider the simpler case of a discrete dynamical system, with coordinates $q_i$, with a Lagrangian $L(q_i, q'_i, t)$ which is invariant under an infinitesimal symmetry transformation $q_i \rightarrow q_i + \epsilon_i(q, t)$, so that the change produced in the Lagrangian is

$$\delta L = \sum_i \left[ \frac{\partial L}{\partial q_i} \epsilon_i(q, t) + \frac{\partial L}{\partial q'_i} \frac{d\epsilon_i(q, t)}{dt} \right]$$

where the time derivative of $\epsilon$ is a stream derivative including the variation of all of the $q$’s, $\frac{d\epsilon_i(q, t)}{dt} = \frac{\partial \epsilon_i(q, t)}{\partial t} + \sum_j \frac{\partial \epsilon_i(q, t)}{\partial q_j} \dot{q}_j$. Using the equations of motion on the first term in $\delta L$, we have

$$\delta L = \sum_i \left[ \left( \frac{d}{dt} \frac{\partial L}{\partial q_i} \right) \epsilon_i(q, t) + \frac{\partial L}{\partial q'_i} \frac{d\epsilon_i(q, t)}{dt} \right] = \frac{d}{dt} \left( \sum_i \frac{\partial L}{\partial q_i} \epsilon_i(q, t) \right).$$

Thus if the change $\delta L$ is zero,

$$Q = \sum_i \frac{\partial L}{\partial q_i} \epsilon_i(q, t) = \sum_i P_i \epsilon_i(q, t)$$

is conserved. Here $P_i$ is the canonical momentum conjugate to $q_i$.

But we see that the quantum mechanical commutator or classical Poisson bracket generates (minus) the infinitesimal symmetry transformation,

$$[Q, q_i] = \sum_j P_j \epsilon_j(q, t, q_i) = \sum_j [P_j, q_i] \epsilon_j(q, t) = \sum_j [-\delta_{ij}] \epsilon_j(q, t)$$

$$= -\epsilon_i(q, t) = -\delta q_i$$

In Field Theory

In field theory the situation is similar but complicated by our consideration of changes in the $x^\mu$ as well as in the fields.

We considered an infinitesimal variation

$$x_\mu \rightarrow x'_\mu = x_\mu + \delta x_\mu$$

with the fields varying by

$$\delta \phi_i(x') = \phi_i(x) + \delta \phi_i(x; \phi_k(x)) = \phi_i(x') + \delta \phi_i.$$  

(2)

The two forms $\delta \phi$ and $\delta \phi$ are useful in differing expressions for $\delta \mathcal{L}$, which we defined by

$$\delta \mathcal{L}(\phi'_i(x'), \partial'_\mu \phi'_i(x'), x') = \mathcal{L}(\phi_i(x'), \partial'_\mu \phi'_i(x'), x) - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x) \left[ \frac{\partial x'_\mu}{\partial x'^\nu} \right].$$

We found that the current for this infinitesimal transformation

$$\epsilon J^\mu = -\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial \phi_i} \partial_\nu \phi_i \delta x'^\nu - \mathcal{L} \delta x'^\mu + \Lambda^\mu.$$  

(3)

has a vanishing divergence if $\delta \mathcal{L} = \partial_\mu \Lambda^\mu$.

Let us avoid some interpretation problems by assuming the transformation doesn’t change time $\delta t = 0$ so we have no problem defining a conserved charge as $Q = \int d^3x J^0(x)$, and let’s assume we don’t need a $\Lambda$. Then

$$Q = -\int d^3x \frac{\partial \mathcal{L}}{\partial \phi_i} (\delta \phi_i(x) + \delta x'^\nu \partial_\nu \phi_i(x)) = -\int d^3x \pi_i(x) \delta \phi_i(x),$$

where $\pi_i(x)$ is the canonical momentum density conjugate to $\phi_i$. Then

$$[Q, \phi_j(\vec{y})] = \int d^3x \left[ \pi_i(x) \delta \phi_i(\vec{x}) \right], \phi_j(\vec{y}) = -\int d^3x \left[ \pi_i(x), \phi_j(\vec{y}) \right] \delta \phi_i(\vec{x})$$

$$= \int d^3x \left[ -i \delta^3(\vec{x} - \vec{y}) \delta_{ij} \right] \phi_i(\vec{x}) = i \delta \phi_j(\vec{y}).$$

So $Q$ does generate the infinitesimal symmetry transformation at each point in space.