Last time we saw that the calculation of the first order (in $\alpha$) correction to $F_2$ was untroubled by infrared or ultraviolet divergences, but the expression for the first order correction to $F_1$,

$$\delta F_1(q^2) = 2ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int dx \, dy \, dz \, \delta(1-x-y-z) \left\{ \frac{-\ell^2 + 2(1-x)(1-y)q^2 + 2(1-4z+z^2)m^2}{(\ell^2 - \Delta + i\epsilon)^3} \right\},$$

(with $\Delta = -xy q^2 + (1-z)^2 m^2$), diverges in the ultraviolet because of the term $\ell^2$ in the numerator, and also, at $q = 0$, because $\Delta = (1-z)^2 m^2$ vanishes in the denominator at the $z \approx 1$ end of the integration interval.

We may regulate the infrared divergence by pretending that the photon has a small mass $\mu$ instead of being massless, thereby changing the photon propagator’s denominator $(k-p)^2 + i\epsilon \rightarrow (k-p)^2 - \mu^2 + i\epsilon$, which changes $\Delta \rightarrow -xy q^2 + (1-z)^2 m^2 + z\mu^2$. To take care of the ultraviolet divergence, pretend that there is also another, very heavy, photon of mass $\Lambda$ with imaginary coupling, so that there is another term, and now the photon propagator

$$\frac{-ig_{\nu\rho}}{(k-p)^2 + i\epsilon} \rightarrow \frac{-ig_{\nu\rho}}{(k-p)^2 - \mu^2 + i\epsilon} - \frac{-ig_{\nu\rho}}{(k-p)^2 - \Lambda^2 + i\epsilon},$$

Eventually we will take $\mu \rightarrow 0$ and $\Lambda \rightarrow \infty$, and in terms without ultraviolet divergences the heavy photon’s contribution will vanish. Using the second and last expressions from page 3 of last time’s notes, this gives

$$\delta F_1(q^2) = \frac{2}{4\pi} \frac{e^2}{4\pi} \int dx \, dy \, dz \, \delta(1-x-y-z) \left[ \ln \frac{-xy q^2 - (1-z)^2 m^2 + z\Lambda^2}{-xy q^2 + (1-z)^2 m^2 + z\mu^2} + \frac{(1-x)(1-y)q^2 + (1-4z+z^2)m^2}{-xy q^2 + (1-z)^2 m^2 + z\mu^2} \right].$$
As we are interested in the $\Lambda \to \infty$ limit, we can drop the other terms in the numerator of the log. For $q^2 = 0$ the integrand is independent of $x$ and $y$ so
\[
\int dx \, dy \, dz \, \delta(1 - x - y - z) \to \int_0^1 dz (1 - z),
\]
and
\[
\delta F_1(0) = \frac{\alpha}{2\pi} \int_0^1 dz (1 - z) \left[ \ln \frac{z \Lambda^2}{(1 - z)^2 m^2 + z \mu^2} + \frac{(1 - 4z + z^2)m^2}{(1 - z)^2 m^2 + z \mu^2} \right],
\]
which is 7.32, and what we will need in explaining how to throw $\delta F_1(0)$ away.

But first we will ask about what is left of $\bar{F}$ after we throw away the troublesome pieces we know ought not make $F_1(0)$ differ from 1. That is, define
\[
\bar{\delta} F_1(q^2) := \lim_{\Lambda \to \infty} \left( F_1(q^2) - F_1(0) \right)
\]
\[
= \frac{\alpha}{2\pi} \int dx \, dy \, dz \, \delta(1 - x - y - z) \left[ \ln \frac{(1 - z)^2 m^2 + z \mu^2}{-xy q^2 + (1 - z)^2 m^2 + z \mu^2} + \frac{(1 - x)(1 - y)q^2 + (1 - 4z + z^2)m^2}{-xy q^2 + (1 - z)^2 m^2 + z \mu^2} - \frac{(1 - 4z + z^2)m^2}{(1 - z)^2 m^2 + z \mu^2} \right].
\]

The logarithm is not singular as $\mu \to 0$ so we can drop those terms, and this term gives $\frac{\alpha}{2\pi} \int_0^1 (1 - z)dz \int_0^1 d\xi \ln \frac{m^2}{-\xi(1-\xi)q^2 + m^2}$, which is nonsingular, where we substituted $x = (1 - z)\xi$.

Now the part of $\bar{\delta} F_1(q^2)$ which does blow up for $\mu^2 \to 0$ comes from the $z = 1, x = y = 0$ endpoint of the integral, so except for vanishing denominators, we can make that substitution, and the same substitution as above, $x = (1 - z)\xi$, and also $w = 1 - z$, to get
\[
\bar{\delta} F_1(q^2) \sim \frac{\alpha}{4\pi} \int_0^1 dz \int_0^{1-z} dx \frac{q^2 - 2m^2}{m^2(1-z)^2 - q^2 x(1-z-x)+\mu^2} - \frac{-2m^2}{m^2(1-z)^2+\mu^2}
\]
\[
= \frac{\alpha}{4\pi} \int_0^1 d(w^2) \int_0^1 d\xi \frac{q^2 - 2m^2}{(m^2 - q^2 \xi(1-\xi))w^2 + \mu^2} - \frac{-2m^2}{m^2w^2 + \mu^2}
\]
\[
= \frac{\alpha}{4\pi} \int_0^1 d\xi \frac{q^2 - 2m^2}{(m^2 - q^2 \xi(1-\xi))w^2 + \mu^2} \ln \left( \frac{(m^2 - q^2 \xi(1-\xi))w^2 \mu^2}{\mu^2} \right) + \frac{\alpha}{2\pi} \ln \left( \frac{m^2}{\mu^2} \right).
\]
Thus
\[
F_1(q^2) = 1 - \frac{\alpha}{2\pi} f(r(q^2)) \ln \left( \frac{m^2}{\mu^2} \right) + \text{IR nonsingular terms},
\]
where
\[ f_{IR}(q^2) = \int_0^1 \left( \frac{m^2 - q^2/2}{m^2 - q^2 \xi (1 - \xi)} \right) d\xi - 1. \]

Read the first third of page 200

So the next order corrections to the elastic scattering amplitude subtracts a piece proportional to the lowest order calculation. But we also saw that the lowest order calculation of the cross section for emission of a soft photon of energy less than \( \varepsilon \) was similarly proportional to the elastic scattering,

\[ d\sigma(\vec{p} \to \vec{p}' + \gamma) = d\sigma(\vec{p} \to \vec{p}') \cdot \frac{\alpha}{2\pi} \ln \left( \frac{\varepsilon^2}{\mu^2} \right) I(\vec{v}, \vec{v}'). \]

On Evaluating \( I(\vec{v}, \vec{v}') \)

The expression for \( I(\vec{v}, \vec{v}') \) is given by

\[ I(\vec{v}, \vec{v}') = \int \frac{d\Omega_k}{4\pi} \frac{2p \cdot p'}{(k \cdot p')(k \cdot p)} - \frac{m^2}{(k \cdot p')^2} - \frac{m^2}{(k \cdot p)^2} \]

which can be evaluated using the Feynman parameter trick. First of all the last two terms in (6.15) can be evaluated, for each choosing the \( z \) access along the velocity, so they contribute

\[ \int d\Omega_k \frac{-2m^2}{4\pi E^2} \frac{1}{(1 - v \cos \theta)^2} = \frac{-m^2}{E^2} \int_{-1}^{1} \frac{du}{(1 - vu)^2} = -\frac{m^2}{E^2 v} \left. \frac{1}{1 - vu} \right|_{-1}^{1} = -\frac{m^2}{E^2 v} \left( \frac{1}{1 - v} - \frac{1}{1 + v} \right) = -\frac{2m^2}{E^2} \frac{1}{1 - v^2} = -2. \]

For the first term, use

\[ \frac{1}{(k \cdot p')(k \cdot p)} = \int_0^1 \frac{d\alpha}{(k \cdot (\alpha p' + (1 - \alpha)p))^2}. \]

Recalling we are working in a frame with \( E' = E \) and \( q^0 = 0 \), this is just like \( 1/2m^2 \) times the above with \( E'v \to \alpha(\vec{p}' - \vec{p}) + \vec{p} = \alpha \vec{q} + \vec{p} \), so the integral is

\[ \frac{1}{E^2 - p^2 - 2\alpha \vec{p} \cdot \vec{q} - \alpha^2 q^2}. \]

As \( \vec{p} \cdot \vec{q} = \vec{p'} \cdot \vec{p}' - \vec{p}^2 = -\frac{1}{2}q^2 = q^2/2 \), we have

\[ \int d\Omega_k \frac{1}{4\pi (k \cdot p')(k \cdot p)} = \int_0^1 d\alpha \frac{1}{m^2 - \alpha (1 - \alpha)q^2}. \]
2p \cdot p' = 2m^2 - q^2 \text{ so all together, }

\mathcal{I}(\vec{v}, \vec{v}') = \int_0^1 \left( \frac{2m^2 - q^2}{m^2 - \alpha(1 - \alpha)q^2} \right) \, d\alpha - 2 =: 2f_m(q^2).

\text{If } -q^2 \gg m^2, \text{ the integral is given by equal contributions near each endpoint, so } 
\approx 2 \int_0^1 d\alpha \frac{1}{\alpha - m^2/q^2} \approx 2 \ln(-q^2/m^2).

\text{Read the bottom third of page 200 and the top half of 201.}

\text{It would be good to at least skim section 6.5 to get the gist of the argument to all orders for infrared behavior.}