Lecture 21 Nov. 15, 2007

$\delta F_1(q^2)$, Källen-Lehmann, Σ_2.

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Last time we saw that the calculation of the first order (in α) correction to F_2 was untroubled by infrared or ultraviolet divergences, but the expression for the first order correction to F_1,

$$\begin{aligned}\delta F_1(q^2) &= 2ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int dx\, dy\, dz\, \delta(1-x-y-z) \\ &\qquad \frac{-\ell^2 + 2(1-x)(1-y)q^2 + 2(1-4z+z^2)m^2}{(\ell^2 - \Delta + i\epsilon)^3},\end{aligned}$$

(with $\Delta = -xy\, q^2 + (1-z)^2 m^2$), diverges in the ultraviolet because of the term ℓ^2 in the numerator, and also, at $q = 0$, because $\Delta = (1-z)^2 m^2$ vanishes in the denominator at the $z \approx 1$ end of the integration interval.

We may regulate the infrared divergence by pretending that the photon has a small mass μ instead of being massless, thereby changing the photon propagator's denominator $(k-p)^2 + i\epsilon \to (k-p)^2 - \mu^2 + i\epsilon$, which changes $\Delta \to -xy\, q^2 + (1-z)^2 m^2 + z\mu^2$. To take care of the ultraviolet divergence, pretend that there is also another, very heavy, photon of mass Λ with imaginary coupling, so that there is another term, and now the photon propagator

$$\frac{-ig_{\nu\rho}}{(k-p)^2 + i\epsilon} \to \frac{-ig_{\nu\rho}}{(k-p)^2 - \mu^2 + i\epsilon} - \frac{-ig_{\nu\rho}}{(k-p)^2 - \Lambda^2 + i\epsilon},$$

Eventually we will take $\mu \to 0$ and $\Lambda \to \infty$, and in terms without ultraviolet divergences the heavy photon's contribution will vanish. Using the second and last expressions from page 3 of last time's notes, this gives

$$\begin{aligned}\delta F_1(q^2) &= \frac{2}{4\pi}\frac{e^2}{4\pi} \int dx\, dy\, dz\, \delta(1-x-y-z) \\ &\qquad \left[\ln \frac{-xy\, q^2 - (1-z)^2 m^2 + z\, \Lambda^2}{-xy\, q^2 + (1-z)^2 m^2 + z\, \mu^2} \right. \\ &\qquad\qquad \left. + \frac{(1-x)(1-y)q^2 + (1-4z+z^2)m^2}{-xy\, q^2 + (1-z)^2 m^2 + z\, \mu^2} \right].\end{aligned}$$

As we are interested in the $\Lambda \to \infty$ limit, we can drop the other terms in the numerator of the log. For $q^2 = 0$ the integrand is independent of x and y so

$\int dx\,dy\,dz\,\delta(1-x-y-z) \to \int_0^1 dz(1-z)$, and

$$\delta F_1(0) = \frac{\alpha}{2\pi}\int_0^1 dz\,(1-z)\left[\ln\frac{z\,\Lambda^2}{(1-z)^2m^2+z\,\mu^2}+\frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\,\mu^2}\right],$$

which is 7.32, and what we will need in explaining how to throw $\delta F_1(0)$ away.

Now we turn to understanding the divergences, to the process of renormalization.

Read sections 7.1 and 7.5.

In section 7.1, we find (7.31):

$$\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz\left[-z\ln\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}+2(1-z)\frac{z(2-z)m^2}{(1-z)^2m^2+z\mu^2}\right]$$

so

$$\delta F_1(0)+\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dz(1-2z)\ln\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}+m^2\frac{(1-z)(1-z^2)}{(1-z)^2m^2+z\mu^2}.$$

In the first term integrate by parts, with $u=z(1-z)$, $v=\ln ...$, with $uv=0$ at both endpoints, and

$$dv = \frac{1}{z}+\frac{2(1-z)m^2-\mu^2}{(1-z)^2m^2+z\mu^2},$$

so

$$\begin{aligned} -\int u dv &= -\int_0^1\left[(1-z)+z(1-z)\frac{2(1-z)m^2-\mu^2}{(1-z)^2m^2+z\mu^2}\right] \\ &= -\int_0^1(1-z)\left[1-1+m^2\frac{1-z^2}{(1-z)^2m^2+z\mu^2}\right], \end{aligned}$$

which cancels the second term, and

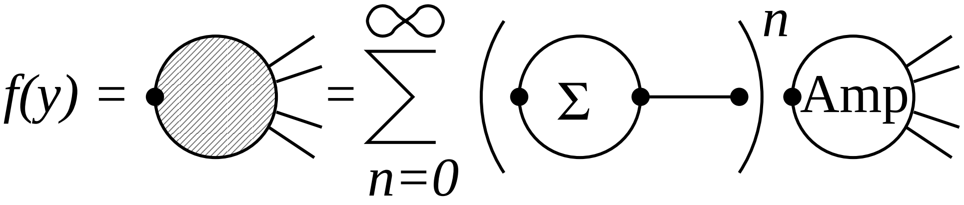
$$\delta F_1(0)+\delta Z_2 = 0.$$

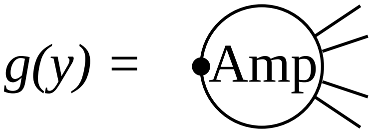
We are going to skip sections 7.2–7.4, but we need to make use of the main result of section 2, which is that the invariant amplitude $\mathcal{M}$ for any process is correctly given by the sum of amputated connected diagrams, but with a factor of $\sqrt{Z}$ for each external line.

A handwaving sketch of the derivation of this fact, given in section 2, is to ask how the fourier transform in x of a time ordered product involving $\phi(x)$ behaves near $p^2 = m^2$, where for simplicity I am taking a scalar field of physical mass m. On the one hand, we know that the time ordered product is given by the sum over *all* diagrams, so we have

$$\langle 0|\, T\phi(x)\ldots|0\rangle = \int dy D(x-y) f(y),$$

where





with $f(y)$ the sum of all diagrams (with the line to x removed) and $g(y)$ is the sum of diagrams with amputation on that leg.

$$\begin{aligned}
\langle 0|\, T\phi(x)\ldots|0\rangle &= \int dy D(x-y) f(y) \\
&= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_0^2 + i\epsilon} e^{-ipx} \tilde{f}(p) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2 - m_0^2 + i\epsilon} \sum_{n=0}^{\infty} \left(-i\Sigma(p^2) \frac{i}{p^2 - m_0^2 + i\epsilon} \right)^n \tilde{g}(p) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\epsilon} \tilde{g}(p)
\end{aligned}$$

The fourier transform will have a pole at $p^2 = m^2 = m_0^2 + \Sigma(p^2)$ and in the vicinity of that pole, we have

$$\begin{aligned}
\langle 0|\, T\phi(x)\ldots|0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2 - m^2 - (p^2 - m^2)\dfrac{d\Sigma(p^2)}{dp^2} + i\epsilon} \tilde{g}(p) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{iZ}{p^2 - m^2 + i\epsilon} \tilde{g}(p),
\end{aligned}$$

where

$$Z^{-1} = 1 - \left.\frac{d\Sigma(p^2)}{dp^2}\right|_{p^2=m^2} .$$

On the other hand, the time ordered product should be

$$\langle 0|\,\phi(x)\,|p\rangle\,\frac{i}{p^2 - m^2 + i\epsilon}\mathcal{M},$$

and $\langle 0|\,\phi(0)\,|p\rangle = \sqrt{Z}$, so the invariant amplitude is given by $\sqrt{Z}\tilde{g}$, that is, the sum of all amputated diagrams with a factor of $\sqrt{Z}$ for each external leg.