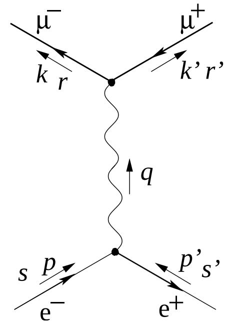
Lecture 17 Nov. 1, 2007

Crossing symmetry; Compton Scattering
Algebra help for Page 161, and more

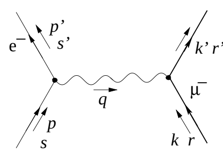
First, we are going to skip sections 5.2 and 5.3. Experimentalists who might someday need to measure and/or calculate polarized cross sections should, and should be able to, read 5.2 on their own.
Last time we calculated $e^- e^+ \to \mu^- \mu^+$, and found

$$\mathcal{M} = ie^2 \bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^r(k)\gamma_\mu v^{r'}(k')/q^2,$$

where $q_\mu = p_\mu + p'_\mu$.



Now we consider $e^- \mu^-$ scattering, which has the same diagram, flipped on its side, with the only changes that 1) we have relabeled the momentum, and 2) the $\bar{v}^{s'}(p')$ for the incoming e^+ is replaced by $\bar{u}^{s'}(p_1)$ for the outgoing electron, and the $v^{r'}(k')$ of the outgoing μ^+ is replaced by $u^r(p_2)$ for the incoming μ^-.



Help for page 161

I am not going to discuss the algebra for the Klein-Nishina formula in lecture, but I want to make sure you can follow it on your own. There are two involved trace calculations not completely spelled out in the book. Here they are:

$$\begin{aligned}
I &= \mathrm{Tr}\left\{(\not p' + m)\left(\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu\right)(\not p + m)\left(\gamma_\nu \not k \gamma_\mu + 2\gamma_\mu p_\nu\right)\right\} && (1)\\
&= \mathrm{Tr}\Big\{(\not p' + m)\Big[\gamma^\mu \not k(-2\not p + 4m)\not k \gamma_\mu + 2(-2\not p \gamma^\nu \not k + 4mk^\nu)p_\nu \\
&\qquad + 2\gamma^\mu(\not p + m)\not p \not k \gamma_\mu + 4p^2(-2\not p + 4m)\Big]\Big\} && (2)\\
&= \mathrm{Tr}\Big\{(\not p' + m)\Big[4\not k \not p \not k + 16mk^2 - 4m^2\not k + 8mk\cdot p + 8mp\cdot k - 4\not k m^2 \\
&\qquad - 8m^2\not p + 16m^3\Big]\Big\} && (3)
\end{aligned}$$

$$\begin{aligned}
&= 32p'\cdot k p\cdot k - 16k^2 p\cdot p' - 16m^2 k\cdot p' - 16m^2 k\cdot p' - 32m^2 p\cdot p' \\
&\qquad + 64m^2k^2 + 64m^2 k\cdot p + 64m^4 && (4)\\
&= 16\left(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p\cdot k' + 2(p\cdot k)(p'\cdot k)\right) && (5)\\
&= 16\Big(4m^4 + m^2(t - 2m^2) + 2m^2(s - m^2) + m^2(u - m^2) \\
&\qquad - \frac{1}{2}(s - m^2)(u - m^2)\Big) && (6)\\
&= 16\left(2m^4 + m^2(s - m^2) - \frac{1}{2}(s - m^2)(u - m^2)\right). && (7)
\end{aligned}$$

In the second line, (2), the first term comes from $\gamma^\nu \not p \gamma_\nu = -2\not p$ (5.9a) and $\gamma^\nu m \gamma_\nu = 4m$; the second term from $\gamma^\mu \not k \gamma^\nu \not p \gamma_\mu = -2\not p \gamma^\nu \not k$ (5.9c); the third just rewrites $2\gamma^\mu p^\nu (\not p + m)\gamma_\nu \not k \gamma_\mu$, and the fourth is $4p^2\gamma^\mu(\not p + m)\gamma_\mu$ and uses the same tricks as the first.

In the third line, (3), the first and second terms come from the first in (2), the first again using 5.9c. The third and fourth come from the second of (2), using $\not p^2 = p^2 = m^2$. In the third term of (2), we use $(\not p + m)\not p = p^2 + m\not p = m(\not p + m)$ so $2\gamma^\mu(\not p + m)\not p \not k \gamma_\mu = 2m\gamma^\mu(\not p + m)\not k \gamma_\mu = -4m^2\not k + 8mp\cdot k$, and of course in the last term we replace p^2 by m^2.

In the fourth expression, (4), the first line comes from the $\not p'$ and the second line from m of the $(\not p' + m)$.

In the fifth line, (5), we use $k^2 = 0$ for a photon, and as $p\cdot k' = p'\cdot k$, this verifies 5.82.

In the sixth line, (6), we use $2p\cdot k = 2p'\cdot k' = s - m^2$, $2k\cdot p' = 2k'\cdot p = m^2 - u$, $2p\cdot p' = 2m^2 - t$, and $2k\cdot k' = -t$ from 5.83. Finally, in (7) we use $s + t + u = \sum m_i^2 = 2m^2$, and so we verify 5.84.

Now for the second term,

$$\begin{aligned}
II &= \mathrm{Tr}\left\{(\not p' + m)\left(\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu\right)(\not p + m)\left(\gamma_\mu \not k' \gamma_\nu - 2\gamma_\nu p_\mu\right)\right\} && (8)\\
&= \mathrm{Tr}\Big\{(\not p' + m)\Big[(-2\not p \gamma^\nu \not k + 4k^\nu m)\not k' \gamma_\nu - 2\not p \not k(-2\not p + 4m) \\
&\qquad + 2(-2\not p + 4m)\not k' \not p - 4m^2(\not p + m)\Big]\Big\} && (9)\\
&= \mathrm{Tr}\Big\{(\not p' + m)\Big[-8k\cdot k' \not p + 4m\not k' \not k + 4\not p \not k \not p - 8m\not p \not k - 4\not p \not k' \not p \\
&\qquad + 8m\not k' \not p - 4m^2(\not p + m)\Big]\Big\} && (10)\\
&= -32k\cdot k' p\cdot p' + 16m^2 k\cdot k' + 16\cdot 2p\cdot p' k\cdot p - 16m^2 k\cdot p' \\
&\qquad - 32m^2 k\cdot p - 16\cdot 2\, p\cdot p' p\cdot k' + 16m^2 k'\cdot p'
\end{aligned}$$

$$\begin{aligned} & +32m^2 k' \cdot p - 16m^2 p \cdot p' - 16m^4 & (11)\\ = & \; 16\left(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)(p'\cdot k)\right) & (12)\\ = & \; -8\left(4m^4 + m^2(s-m^2) + m^2(u-m^2)\right) & (13)\end{aligned}$$

and so we verify 5.86.

Note on 5.88: recall from freshman physics that $\lambda' - \lambda = \frac{h}{mc}\left(1-\cos\theta\right)$. From 5.88, 5.87 gives

$$\begin{aligned}\frac{1}{4}\sum\left|\mathcal{M}^2\right| &= 2e^4\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 2(1-\cos\theta) + (1-\cos\theta)^2\right)\\ &= 2e^4\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right).\end{aligned}$$

In the third line of 5.90, they use $\vec{p}\,' = \vec{\omega} - \vec{k}\,' = (-\omega'\sin\theta, 0, \omega - \omega'\cos\theta)$, so $E' = \sqrt{m^2 + \omega^2 + (\omega')^2 - 2\omega\omega'\cos\theta}$. In the fifth line we note that as $E' + \omega' = m + \omega$, $E' + \omega' - \omega\cos\theta = m + \omega - \omega\cos\theta$.

In 5.93 you might wonder why the last term isn't even more singular, but in fact $1/(p\cdot k) - 1/(p\cdot k') \to -2/m^2$ at $\theta = \pi$, so it doesn't blow up.

In the expression below 5.93, the intermediate expression is

$$\frac{2\pi\alpha^2}{4E(E+\omega\cos\theta)} \approx \frac{2\pi\alpha^2}{4E(\omega + \frac{m^2}{2\omega} + \omega\cos\theta)}$$

which should explain the last expression. The total cross section is more straightforwardly evaluated as

$$\sigma = \frac{2\pi\alpha^2}{4E\omega}\int_{-1}^{1} d(\cos\theta)\frac{1}{1+\cos\theta+\frac{m^2}{2\omega^2}} = \frac{2\pi\alpha^2}{4E\omega}\ln\left(1+\frac{4\omega^2}{m^2}\right) \approx \frac{2\pi\alpha^2}{s}\ln\left(\frac{s}{m^2}\right).$$

Here is an alternative explanation to what is given in the bottom of p165 and most of p166:

To see why at high energy 5.97 does not produce a more severe singularity as $m \to 0$ and $\theta = \pi$, recall that the helicity of a particle is projected by

$$\frac{1}{2}(1\pm\gamma_5) = \begin{pmatrix} 1\!\!1_{2\times 2} & 0\\ 0 & 0\end{pmatrix} \quad\text{or}\quad \begin{pmatrix} 0 & 0\\ 0 & 1\!\!1_{2\times 2}\end{pmatrix}$$

in our representation. Helicity is preserved by the combination of γ^ν from the vertex and the $\not{p} - \not{k}$ from the propagator. At very high energies, the dominant

term is the one which preserves helicity for the electron, and therefore in the backwards scattering its spin changes by $\pm\hbar$ in the z direction. The spin of the photon moving in the $\pm z$ direction can only be $\pm\hbar$, so it cannot change by ± 1, so angular momentum L_z must pick up one unit, but that cannot be if all particles are moving in the $\pm z$ direction. thus the scattering amplitude vanishes for $\theta = \pi$ exactly in the massless limit.