Let us remind ourselves that

\[ \psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( a^s_p \epsilon^s(p) e^{-ipx} + b^s_{-p} \epsilon^s(p) e^{ipx} \right) \]

\[ \tilde{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( b^s_p \epsilon^s(p) e^{-ipx} + a^s_{-p} \epsilon^s(p) e^{ipx} \right) \]

\[ u^s(p) = \left( \frac{\sqrt{p \cdot \sigma \xi^s}}{\sqrt{p \cdot \sigma \xi^s}} \right) \]

\[ v^s(p) = \left( \frac{\sqrt{p \cdot \sigma \eta^s}}{-\sqrt{p \cdot \sigma \eta^s}} \right) \]

For 3.129-3.132, we note more generally that

\[ P \psi(t, \vec{x}) \Gamma \psi(t, \vec{x}) P^{-1} = \tilde{\psi}(t, -\vec{x}) \gamma^0 \Gamma \gamma^0 \psi(t, -\vec{x}). \]

so each spatial index gives \(-1\) and \(\gamma^5\) gives \(-1\).

For a positronium atom at rest in a state of angular moment \( \ell \), we may write

\[ |\psi_{\ell m}\rangle = \int_0^\infty \rho^2 dp \int_0^\pi \sin \theta \int_0^{2\pi} d\phi f(p) Y_{\ell m}^* (\theta, \phi) a^{-\dagger}_{-\rho} b_{-\rho}^\dagger |0\rangle, \]

so

\[ P |\psi_{\ell m}\rangle = \eta_\rho \eta_\bar{\rho} \int_0^\infty \rho^2 dp \int_0^\pi \sin \theta \int_0^{2\pi} d\phi f(p) Y_{\ell m} (\theta, \phi) a^{-\dagger}_{-\rho} b_{-\rho}^\dagger |0\rangle \]

\[ = (-1)^\ell \int_0^\infty \rho^2 dp \int_0^\pi \sin \theta' \int_0^{2\pi} d\phi' f(p') Y_{\ell m} (\pi - \theta', \phi + \pi) a^{-\dagger}_{-\rho} b_{-\rho}^\dagger |0\rangle \]

\[ = (-1)^{\ell+1} |\psi_{\ell m}\rangle. \]

where we replaced \( \bar{\rho} \) with \( \bar{\rho}' = -\bar{\rho} \), and noted that reversing the direction in the \( Y_{\ell m}^* (\theta \to \pi - \theta, \phi \to \phi + \pi) \) just multiplies by \((-1)^\ell\).

If \( \hat{n} \) is a unit vector in the \( \theta, \phi \) direction,

\[ \hat{n} \cdot \sigma = \left( \begin{array}{cc} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{array} \right) \]
Then
\[ \hat{n} \cdot \vec{\sigma} \left( \begin{array}{c} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{array} \right) = \left( \begin{array}{c} \cos(\theta/2) (\sin \theta \cos(\theta/2) - \cos \theta \sin(\theta/2)) \\ \sin(\theta/2) e^{i\phi} (\sin \theta \cos(\theta/2) + \cos \theta \sin(\theta/2)) \end{array} \right), \]

verifying that \( \xi(\uparrow) \) given at the top of page 68 is an eigenvector of spin in the \( \hat{n} \) direction. What is being done in the middle of p. 68 is choosing the \( \theta/2 \) of Eq. 3.62 for antiparticles to be \( \xi^{-s} \), in a way which allows for an arbitrary \( \hat{n} \).

An aside:

The somewhat arbitrary-looking definition \( \xi^{-s} = -i\sigma_2 (\xi^s)^* \) comes from the way the complex conjugate of the canonical spin \( \frac{1}{2} \) representation of rotations transforms. For any representation \( M \) of any group, the complex conjugate \( M^* \) is also a representation. For example, for SU(3) the antiquarks are the complex conjugate representation from the quarks, and these representations are not equivalent. For SU(2), there is only one representation, up to equivalence, for each dimensionality, so the complex conjugate of a spinor must be equivalent to a spinor, and \( M^* \) must be equivalent to \( M \). That is, there exists a single matrix \( U \) such that \( M^*(g) = U M(g) U^{-1} \) for all elements \( g \in G \) in the group. Such a representation is called a \textbf{real representation}, but it does not mean the elements of \( M \) are real.

For a spinor, a rotation by \( \vec{\omega} \) is represented by \( M(\vec{\omega}) = e^{-i\omega_i \sigma_i / 2} \) so \( M^*(\vec{\omega}) = e^{i\omega_i \sigma_i / 2} = (-i\sigma_2) M(\vec{\omega})(-i\sigma_2)^{-1} \). So if \( \xi^s \), \( (s=1,2) \), transform by the canonical spin \( \frac{1}{2} \) representation, so do \( -i\sigma_2 (\xi^s)^* \).

A note on 3.145:

The difference between \( ^* \) and \( ^\dagger \) is a bit confused here. I would have left off the \( -i \gamma^2 \psi^*(x) \) expression and just noted that in \( (\psi^\dagger)^\dagger \) the transpose acts only in the four dimensional spinor space, not in the full Hilbert space in which \( \psi \) acts. The \( ^\dagger \) does act in the full Hilbert space.

For the general bilinear,
\[
C \bar{\psi} \Gamma \psi C^{-1} = C \bar{\psi} a C^{-1} \Gamma_{ab} C \bar{\psi} b C^{-1} = \\
= (-i\gamma^0 \gamma^2 \psi a) \Gamma_{ab} (-i \bar{\psi} \gamma^0 \gamma^2 b) \\
= (-i)^2 (-1) \bar{\psi} \gamma^0 \gamma^2 \Gamma^T \gamma^0 \gamma^2 \psi \\
= \bar{\psi} \left( \gamma^0 \gamma^2 \right)^{-1} \Gamma^T \gamma^0 \gamma^2 \psi
\]

Now \( (\gamma^0 \gamma^2)^{-1} \gamma^\mu \gamma^0 \gamma^2 = -\gamma^\mu T \), because under transpose \( \gamma^\mu \) changes sign for \( \mu = 1, 3 \) and doesn’t for \( \mu = 0, 2 \), while the opposite occurs under commutation with \( \gamma^0 \gamma^2 \). Thus if \( \Gamma \) is built of \( \gamma^\mu \)'s, each one contributes a minus sign, but also the order of the indices is reversed, which gives a minus sign if the number of \( \gamma^\mu \)'s with different \( \mu \)'s is 2 or 3 mod 4. Thus
\[
C \bar{\psi} \Gamma \psi C^{-1} = \bar{\psi} \psi \text{ for } \psi, \psi \gamma_5 \psi, \psi \gamma^0 \gamma_5 \psi, \\
C \bar{\psi} \Gamma \psi C^{-1} = -\bar{\psi} \Gamma \psi \text{ for } \psi \gamma^0 \psi, \psi \gamma^0 \gamma_5 \psi.
\]

Please read pages 65-71.