Little Note on Fierz

What is \((\sigma^\mu)_{\alpha\beta} (\sigma_\mu)_{\gamma\zeta}\)?

If \(\alpha = \beta\), only \(\sigma^0 = \delta_{\alpha\beta}\) and \(\sigma^3 = \delta_{\alpha\beta}(-1)^{\alpha+1}\) contribute, giving

\[\delta_{\alpha\beta} \delta_{\gamma\zeta} \left( 1 - (-1)^{\alpha+\gamma} \right) = 2\delta_{\alpha\beta} \delta_{\gamma\zeta} (1 - \delta_{\alpha\gamma}) = 2\epsilon_{\alpha\gamma} \epsilon_{\beta\zeta} \quad \text{for} \quad \alpha = \beta.\]

If \(\alpha \neq \beta\), only \(\sigma^1\) and \(\sigma^2\) can contribute, in which case only \(\gamma \neq \zeta\) gives nonzero, and then

- either \(\alpha = \gamma\), and the two terms cancel, \((\sigma^1_{\alpha\beta})^2 + (\sigma^2_{\alpha\beta})^2 = 0\),
- or \(\alpha = \zeta\), \(\beta = \gamma\) \(\sigma^1_{\alpha\beta} \sigma^1_{\gamma\zeta} = 1\), \(\sigma^2_{\alpha\beta} \sigma^2_{\gamma\zeta} = 1\), so we get \(-2\).

Thus when \(\alpha \neq \beta\), \((\sigma^\mu)_{\alpha\beta} (\sigma_\mu)_{\gamma\zeta} = 2\epsilon_{\alpha\gamma} \epsilon_{\beta\zeta}\).

So in either case, \((\sigma^\mu)_{\alpha\beta} (\sigma_\mu)_{\gamma\zeta} = 2\epsilon_{\alpha\gamma} \epsilon_{\beta\zeta}\).

Another Approach

Consider the mapping of \(2 \times 2\) matrices

\[M \rightarrow \sum_\mu \sigma^\mu M \sigma_\mu.\]

This is a linear real transformation, so we may describe it by its action on a basis of \(2 \times 2\) matrices, in particular using the identity and the Pauli matrices:

\[\mathbb{I} \rightarrow (\sigma^0)^2 - \sum_{j=1}^3 (\sigma^j)^2 = -2 \mathbb{I}\]

\[\sigma^j \rightarrow \sigma^0 \sigma^j \sigma^0 - \sum_{k=1}^3 \sigma^k \sigma^j \sigma^k = \sigma^j + \sum_{k=1}^3 \left[ -2\delta_{jk} \sigma^k + \sigma^j (\sigma^k)^2 \right] = (1 - 2 + 3)\sigma^j = 2\sigma^j\]

Thus

\[\sum_{\mu=0}^3 p_\mu \sigma^\mu \rightarrow -2 \sum_{\mu=0}^3 p_\mu \sigma_\mu,\]

which is strange, but reverses \(\sigma_L\) and \(\sigma_R\), in addition to multiplying by \(-2\).