1) In lecture we defined the Pauli-Lubanski vector
\[ W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu L_{\rho\sigma}, \]
built of the generators of the Poincaré group. Show that
(a) \[ [W^\mu, P_\nu] = 0, \]
find \[ [W^\mu, L_{\nu\rho}] \], and then show \( W^2 \) is a Casimir operator of the Poincaré group, that is
(b) \[ [W^\mu W^\nu, P_\rho] = 0, \]
(c) \[ [W^\mu W^\nu, L_{\alpha\beta}] = 0. \]
Also show, if you haven’t already, that \( P^2 \) is also a Casimir operator of the group.

It may be useful to examine whether your evaluation of \( [L_{\alpha\beta}, W^\mu] \) is what you would expect for a general vector, \( [L_{\alpha\beta}, V^\mu] \), and also to show that the vector properties implied by the indices do transform correctly under commutator with \( L_{\alpha\beta} \). That is, \( V^\mu F_\mu \) should commute, and the c-numbers \( g_{\mu\nu} \) and \( \epsilon_{\mu\nu\rho\sigma} \), which of course do commute with all operators, should commute with Lorentz transformations despite having Lorentz indices.

2) Consider a single real free scalar field with the Klein-Gordon Lagrangian.

(a) Find the expression for \( T^{\mu\nu} \) in terms of \( \phi, \pi, \) and \( \nabla \phi \). From this plugged into \( \mathcal{M}^{\mu\nu\rho} \), find
(b) \( \vec{J}(t) \)
(c) \( \vec{K}(t) \)
as integrals over products of \( \phi(\vec{x}) \) and \( \pi(\vec{x}) \). [Recall \( \vec{J} = (L_{23}, L_{31}, L_{12}) \) and \( \vec{K} = (L_{01}, L_{02}, L_{03}) \).] From these results, find the values of
(d) \( [P^\mu(t), \phi(\vec{x}, t)] \),
(e) \( [\vec{J}(t), \phi(\vec{x}, t)] \)
(f) \( [\vec{K}(t), \phi(\vec{x}, t)] \)
at equal times, in terms of \( \phi(\vec{x}) \) and its derivatives.