\[ U_0 = m^2 \sum_{i \neq j} - \ln |z_i - z_j| \]

Coulomb Interaction
Energy

\[ m \left( \frac{1}{4} \sum_{j} |z_j|^4 \right) = m \sum_j \phi(z_j) \]

Potential from constant
background charge \( \phi_0 = -\frac{1}{2\pi \epsilon_0} \)

\[ -\nabla^2 \left( \frac{1}{4} |z_j|^4 \right) = -\left( \nabla_x^2 + \nabla_y^2 \right) \left( \frac{1}{4} \frac{(x^2 + y^2)}{e^2} \right) = -\frac{1}{e^2} = 2\pi \phi_0 \]

\[ \Rightarrow \phi_0 = -\frac{1}{2\pi \epsilon_0} \]

\[ -m^2 \ln |z_i - z_j| \]

Neutrality \( \Leftrightarrow \) Largest \( |4| \)

\[ n m + \phi_0 = 0 \]

\[ n = \frac{1}{m} \left( \frac{1}{2\pi \epsilon_0} \right) \]

\[ = \frac{1}{m} \left( \frac{N\phi}{A} \right) \]

\[ 2\pi \epsilon_0 \phi_0 = \phi \]

\[ \Rightarrow \frac{1}{2\pi \epsilon_0} = \frac{\psi}{\psi_0} = \frac{\phi}{\phi_0} = \frac{N\phi}{A} \]

\[ n = 5 \]
\[ \psi_m = \prod_{i<j} (z_i - z_j)^m \ e^{-\frac{1}{4} \sum |z_j|} \]

\[ |\psi_m|^2 = e^{-\frac{2}{m} V_{el}[\mathbf{z}]} \]

\[ V_{el} = m^2 \sum - \hbar \sum |z_i - z_j| + m \sum \frac{|z_j|^2}{4} \]

\[ \rho_e = \frac{1}{m} \left( \frac{N \phi}{A} \right) \quad \text{1st filled LL} \]

\[ -\frac{1}{2\pi} \ e^2 = -\frac{N \phi}{A} \]

\[ \psi_{\text{hole}} = \prod (z_i - Z) \ \psi_m \]

"Anyon"

Fractionalized particle

- \( e^* = \frac{e}{3} \)

Fractional statistics
Hole in the LLL

Remove $e^-$ from $m=0$ state at origin

$$f_{2\text{hole}} = \begin{vmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \end{vmatrix} = z_1 z_2^2 - z_2 z_1^2 = z_1 z_2(z_2 - z_1)$$

$2$ zeroes at the origin

$$\psi_{\text{hole}}[z] = \left( \prod_i^{n} z_i \right) \psi_1[z]$$

$$\psi_{\text{hole}}[z, z] = \prod_i^n (z_i - z) \psi_1(z)$$

$$|\psi_{\text{hole}}|^2 = e^{-\beta U_{\text{classical}}}$$

$$R = \frac{2}{m} \quad U_c^i = -\frac{m}{2} |n\{4\}|^2$$

$$U_{\text{class}} = \sum_{i\neq j} \frac{-e}{|z_i - z_j|} + \frac{1}{4} \sum_j |z_j|^2 + \sum_j \frac{-e}{|z_j - z|}$$

Perfect screening in Plasma

$\Rightarrow$ Precisely one electron

Removed from vicinity of zero.

$\Rightarrow$ HOLE CHARGE +e

$$\psi_{\text{Many holes}} = \prod_{k=1}^M \prod_i (z_i - z_k) \psi_1(z)$$
Each particle sees an $m^\text{th}$-fold zero ("$m$ fluxes") at every other particle $\Rightarrow$ good correlations, smaller Coulomb energy.

- For antisymmetry $m$ must be odd.
- Plasma analogy tells us $n = \frac{1}{m} \frac{1}{2\pi R^2} = \frac{1}{m} \times \text{density of mag flux}$
- LL filling is $\nu = \frac{1}{m} = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \quad = \frac{1}{m} \left( \frac{N_S}{\hbar} \right)$

No adjustable parameters (other than $m$), yet almost exact for almost any realistic repulsion.

Consider

$$V = \sum_{m=0}^{\infty} \sum_{i>j} V_{m'} P_{m'}(i,j)$$

But $P_{m'}(i,j)$ & $P_{m''}(k,l)$ don't commute.
Hard core potential: \( V_{m'1} = 0 \) for all \( m' \geq m \), then since

\[
P_{m'}(i,j)\psi_m = 0 \quad (m' < m)
\]

\[\Rightarrow \quad V\psi_m(z) = 0 \quad \text{e.g. } m = 3 \quad V = V_0 p_0 + V_1 p_1 + V_2 p_2 \quad \text{EXACT EIGENSTATE}
\]

Since the angular momenta of pairs can only change discretely, the Laughlin ground state has a gap.

E.g. \( m = 3 \), any excitation weakens the correlations by forcing at least one pair of particles to have relative angular momentum 1, rather than 3. \( \Delta E \sim V_1 \).

**INCOMPRESSIBLE** \( \Rightarrow \) no density fluctuations.

\[
U(r) = -m^2 \ln (r)/r_0
\]

\[
\nabla^2 U = -2\pi m^2 \delta(r) \quad \Rightarrow \quad U(q) = \frac{2\pi m^2}{q^2}
\]

\[
-q^2 U_q = -2\pi m^2
\]

\[
U_{\text{class}} = \frac{1}{2L^2} \sum_{q \neq 0} 8q \frac{2\pi m^2}{q^2} q - q
\]

\[\Rightarrow \quad \langle q q^2 \rangle = L^2 \frac{q^2}{4\pi} \quad \text{Severely suppressed at long wavelengths.}
\]
Short distances, convenient to discuss

\[ g(z) = \frac{(L^2)^2}{\langle\Psi|\Psi\rangle} \int d^2z_3 \ldots d^2z_N |\Phi(0, z, z_3 \ldots z_N)|^2 \]

\[ \lim_{g(z \to \infty)} = 1. \]

Note

\[ g(z) = \frac{1}{n^2} \langle\Psi_g|\Psi(z)\Psi(0)|\Phi_g\rangle \]

\[ m=1 \quad g(z) = \frac{1}{n^2} \int \left[ \left\langle \Psi(z)\Psi(0)\right\rangle \left( \Psi(0)\Psi(z) \right) \right] \]

\[ = \frac{1}{n^2} \left[ \left\langle \Psi(z)\Psi(z)\right\rangle - \left\langle \Psi(0)\Psi(z)\right\langle\Psi(z)\Psi(0)\right\rangle \right] \]

\[ = 1 - \frac{1}{n^2} \left| \sum_{\ell} \Psi(\ell) \Psi^*(\ell) \right|^2 \]

\[ = 1 - e^{-\frac{1}{2}|z|^2} \]

\[ m > 1 \quad g(z) \sim |z|^{2m} \]

\[ (g-1) \]

\[ \left\{ \begin{array}{c}
  \text{m=1} \\
  \text{m=3} \\
  \text{m=1} \\
  \text{m=3} \\
  \text{m=1} \\
  \text{m=3} \\
  \text{m=1} \\
  \text{m=3} \\
\end{array} \right. \]
\[
\frac{\langle \Psi | v | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{nN}{2} \int d^2z \, \frac{e^2}{|z|} \left( g(z) - 1 \right)
\]

\[
V = \frac{1}{2} \int \frac{e^2}{e_i} \left( \Phi_i - \Phi_e \right) \left( \Phi_i - \Phi_e \right)
\]

\[
\frac{V_3}{N} = (-0.4100 \pm 0.0001) \, \frac{e^2}{e_i}
\]

\[
\frac{V_5}{N} = (-0.3277 \pm 0.0001) \, \frac{e^2}{e_i}
\]

16.6 QUASIPARTICLES OF THE LAUGHLIN STATE

\[
\Psi_{\pm}^{(z)}(z) = \left[ \prod_{j=1}^{N} \left( z - z_j \right) \right] \Psi_m(z) \pm e_l m
\]

only on polynomial part \((2 \frac{\partial}{\partial z} + \frac{e_i}{z})\)
\[ |\Psi_2^+|^2 = e^{-\beta u e} e^{-\beta v} \quad e^{2 \rho_n |z_j - z_l|} \]

\[ V = m \sum_{j=1}^{2} (\rho_n |z_j - z_l|) \]

But this means that an electron is physically repelled from the hole \( \Rightarrow \)

\[ e^* = \frac{e}{m} \]

Positive fractional charge!
Does the charge $e^*$ depend on details of the wavefunction? The value of $e^*$ is tied to the value of $\sigma_{xy} = (e^*/\hbar)^1/m$.

$$\gamma_{\alpha} \rightarrow \gamma_{\alpha} e^{-zN/\hbar}$$

Suppose we adiabatically insert a flux at position $Z$, ramping up the flux to $\Phi_0 = \hbar/e$. After this process, the final Hamiltonian is equivalent to the string $\mathcal{H}$, since a unit flux can be gauged away. But, the system does not return to its ground state, instead an anyon is created!

$$\oint E \cdot d\ell = -\frac{\partial \Phi}{\partial t}$$

clockwise

The azimuthal electric field will create a radial current...
Discussion on Fractional Charge

- "Imposter fractional charge"

\[
\psi_k = \frac{1}{\sqrt{3}} \sum_{j=1}^{3} e^{ikj} |lj\rangle
\]

\[
\epsilon_{k\alpha} = -E_{15} - 2J \cos k\alpha
\]

\[
l_{k\alpha} = \frac{2\pi \alpha}{3} \quad (\alpha = 0, 1, 2)
\]
\[ P_n = |n\rangle \langle n| \text{ projection onto state } n \quad (n=1, 2, 3) \]

\[ \langle \Psi_k | P_n | \Psi_k \rangle = \frac{1}{3} \]

\[ Q_n = -eP_n \quad \therefore \quad \langle Q_n \rangle = -\frac{e}{3} \]

However

\[ \langle P_n^2 \rangle - \langle P_n \rangle^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9} \]

\[ \therefore \sqrt{\langle Q_n^2 \rangle - \langle Q_n \rangle^2} = \frac{\sqrt{2}}{3} e \]

Fluctuating charge.

\[ \frac{1}{3} \text{ line we measure } \theta = e, \quad \frac{2}{3} \text{ line } \theta = 0. \]

Characteristic line for the fluctuations \( T \sim \frac{t}{\Delta e} \).

As we separate the sites, the states become degenerate.
The fluctuations are easy to observe.

**Laughlin \( \gamma = 1/3 \) State.**

The QP can be separated without destroying the gap.

\[
\left| x_1, x_2, x_3 \right> \quad \text{No "quasidegeneracy".}
\]

The state is uniquely specified by the position of the Anyons.

"**Emergent Particles**" (elementary).

The charge operator, projected into the low
energy manifold is a sharp operator.

In the three site example, if one introduces the projected charge operator

\[ P^{(n)}_\alpha = P^{n_2}P_nP^{n_2} \]

where

\[ P^{n_2} = \Theta(\Omega - (\hat{H} - E_0)) \quad \text{Finite if } \Delta E < \Omega. \]

then

\[ P^{(n)}_\alpha = \sum \langle \Psi_{k\alpha} | \Theta(\Omega - (\varepsilon_{k\alpha} - \varepsilon_{k0})) | \Psi_{k\alpha} \rangle \]

for \( \alpha = 1, 2, 3 \)

if \( \Omega < \Delta = 3J \), then

\[ P^{(n)}_\alpha = |\Psi_{k\alpha}\rangle \langle \Psi_{k\alpha}| \]
Now \[ P_n^{(s^2)} = 14_{k0} > \frac{1}{3} < 4_{k0} | \]

and

\[
\langle 4_{k0} | (p_n^{(n)})^2 | 4_{k0} \rangle - |\langle 4_{k0} | P_n^{(n)} | 4_{k0} \rangle|^2 = 0
\]

\[
\frac{1}{9} - \frac{1}{3} \cdot \frac{1}{3}.
\]
Flatland cosmologists speculate:

Flatland experimentalists call for the creation of a NATIONAL ACCELERATOR FACILITY

$E_{cm} = 10K$

Virtual electrons
- Charge exciton gap: \( \Delta = \Delta_+ + \Delta_- \)
  - only produced in pairs.

- Conductivity

- \( \Delta_+ = \Delta_- \)

- Noise

- \( S_I(\omega) = e^* \langle I_B \rangle \)