Notice that \( X_n = k_n e^2 = n \left( \frac{2\pi}{L_y} \right) e^2 \ll \sqrt{n} \ell. \)

In the \( n \)th Landau level there are

\[
N = \sum_{k} 1 = L_y \int_{-\infty}^{L_x/e^2} \frac{dk}{2\pi} = \frac{L_x L_y}{2\pi e^2} = \frac{A e B}{2\pi k} = \frac{AB}{(\hbar/e)}
\]

\[
N = \left( \frac{AB}{\Phi_0} \right) = \frac{\Phi}{\Phi_0} = N_{\Phi}
\]

For \( \nu \) levels \( N = \nu N_{\Phi} \).
Dynamics.

\[
\Psi(r,t) = \frac{L_y}{2\pi} \sum \int \frac{dk}{2\pi} a_n(k) \Psi_{nk}(r) e^{-i(n+\frac{1}{2})\omega_{ct} t}
\]

\[\Psi(r^2, t + \frac{2\pi}{ce}) = -\Psi(r,t)\]

Semiclassical motion.

Current flowing in single Landau level state \( \Psi_{kn} \)?

Warm up problem for the case with an electric field.

Heuristically \( I_y = -eV_y = -e \frac{dE}{k} \frac{dk}{dE} = 0 \)
Microscopically:

\[ \psi_k = \frac{1}{\sqrt{\pi^l e^L y}} e^{iky} e^{-\frac{1}{2e} (x-ke^2)} \]

\[ \langle J_y(x,y) \rangle = -\frac{e}{\pi} \frac{\psi_k^*(x,y)}{\psi_k(x,y)} \left[ e^{ik(y+eA_y)} \right] \psi_k(x,y) \]

\[ \langle I_y \rangle = \int_0^{L_x} \langle J_y(x,y) \rangle \, dx \]

\[ = -\frac{e}{\pi} \frac{1}{\sqrt{\pi^l e^L y}} \int dx \, e^{-\frac{1}{2e} (x-ke^2)^2} \left( ke - eBx \right) - eB(x-ke^2) \]

\[ \langle I_y \rangle = e \omega e \frac{1}{\sqrt{\pi^l} e^L y} \int dx \, e^{-\frac{1}{2e} (x-ke^2)^2} (x-ke^2) = 0. \]

As expected
Current in a Field

\[ \vec{E} = [E_x, 0, 0] \]

\[ U(x) = eE_x \]

\[ H = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 (x - k\ell^2)^2 + eE_x. \]

\[ = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 \left(x^2 - 2k\ell^2 x + (k\ell^2)^2\right) + eE_x \]

\[ = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 \left(x^2 - 2\left(k\ell^2 - \frac{eE}{m \omega_c^2}\right)x + \left(k\ell^2 - \frac{eE}{m \omega_c^2}\right)^2\right) \]

\[ + \frac{1}{2} m \omega_c^2 \left((k\ell^2)^2 - (k\ell^2 - \frac{eE}{m \omega_c^2})^2\right) \]

\[ = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 \left(x - X_k'\right)^2 + \frac{1}{2} m \omega_c^2 \left(X_k' - X_k''\right)^2 \]

\[ X_k' = k\ell^2 - \frac{eE}{m \omega_c^2} = X_k - \frac{eE}{m \omega_c^2} \]

\[ \vec{V} = \left( \frac{E}{B} \right) \]

\[ \frac{1}{2} m \omega_c^2 \left(2X_k' + \frac{eE}{m \omega_c^2}\right) = eE X_k' + \frac{1}{2} m \left(\frac{E}{B}\right)^2 \]

\[ H = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 \left(x - X_k'\right)^2 + eE X_k' + \frac{1}{2} m \vec{V}^2 \]
\[ E_{kn} = \hbar v_c \left( n + \frac{1}{2} \right) + eE X_k' + \frac{1}{2} m v^2 \]

\[ v_y = \frac{1}{\hbar} \frac{\partial E_{kn}}{\partial k} = \frac{1}{\hbar} \frac{1}{2} \left( eE k e^2 \right) = \frac{eE e^2}{\hbar} = \left( \frac{E}{B} \right) = \tilde{v} \]

**Physically**

\[ I_y = -n e v_y L_x = -\left( \frac{e}{L_x L_y} \right) L_x \tilde{V} = -e \tilde{V} \]

**Microscopically:**

\[ I_y = -\frac{e}{m} \frac{1}{\hbar m e L_y L_x} \int dx e^{-\frac{1}{e^2} (X - X_k')^2} \left\{ \int dk e B (x) \right\} \]

\[ = \frac{1}{\hbar} \frac{1}{\hbar m e L_y L_x} \int dx e^{-\frac{1}{e^2} (X - X_k')} \left\{ \frac{1}{L_y} \left[ e B (x - X_k') - e B (ke^2 - \frac{e E}{m v_c}) \right] \right\} \]

\[ I_y = -\frac{e v_c}{L_y} \left( \frac{E m}{B e^2} \right) = -\frac{e}{L_y} \tilde{V} \]
Total current in a Landau level is then,

\[ I_y = -\frac{e}{L_y} \int \frac{dk}{(2\pi)} \frac{1}{k} \frac{dE_{kn}}{dk} \]

\[ = -\frac{e}{\hbar} \int dk \frac{dE_{kn}}{dk} = -\frac{e}{\hbar} [M_R - M_L] \]

\[ eV_H = -e(V_R - V_L) = (M_R - M_L) \]

\[ I_y = \frac{e^2}{\hbar} \langle \psi \rangle V_H \]

\[ \sigma_{xy} = \frac{e^2}{\hbar} \psi \]

\[ \sigma_{xy} = 0 \]

\[ \mathcal{G}_{xy} = \frac{\hbar}{e^2 \psi} \]

Edge-states run along equipotentials.
Laughlin's Flux Argument

\[ \Delta \phi = \frac{\hbar}{e} \]

\[ I = \frac{\partial u}{\partial \phi} = \frac{\partial u}{L \partial A} = \frac{\Delta u}{\Delta \phi} \]

\[ \Delta \phi = \frac{\hbar}{e} \psi \rightarrow \psi e^{ieAx} \]

\[ A = \eta \left( \frac{\hbar}{e} \right) \frac{1}{L} \]

No change to energies of bulk

\[ \psi_{nk} = e^{iky} \phi_n(x-x_0) \]

\[ x_0 = \frac{1}{\omega_c} \left( \frac{\hbar k}{m} - \frac{E}{B} + \frac{eA}{m} \right) \]

No change to bulk energies

\[ x_0 \rightarrow x_0 + A \frac{A}{B} \]

Transfering charge from left to right

\[ \Delta U = ne \Delta V \]

\[ \Delta \phi = \frac{h}{e} \]

\[ I_n = \frac{ne \Delta V}{(h/e)} \frac{\Delta V}{h} \]

\[ \Rightarrow \text{INTEGER } \sigma_{xy}. \]
**Percolation Picture**

- Disorder is essential for the IQHE
- Require a gap in the excitation spectrum

![Diagram](image)
Percolation will occur if

\[ M_n^* = \left( n + \frac{1}{2} \right) \hbar \omega_c \]

When the shoreline percolates, the orbital period diverges + the excitation gap vanishes, so that dissipation takes place at the transition between plateaux.
5. **Fractional Quantum Hall Effect**

- Early expts showed that for \( v < 1 \), \( \sigma_{xy} \) & \( \sigma_{xx} \) vanished, as expected from the perspective of the IQHE. However, with the invention of modulation doping in GaAs quantum wells, much higher quality samples were possible, with much less disorder.

- A Wigner crystal was expected for low enough \( e^- \) concentrations in pure samples. Pinned by disorder it would be insulating.

- 1982 Tsui, Störmer & Gossard discovered a QH plateau at \( v = \frac{1}{3} \), with \( \sigma_{xx} \to 0 \) & \( \sigma_{xy} = \frac{1}{3} \frac{e^2}{h} \).

- Tsui joked that it might be quarks! The effect did not involve quarks, but incredibly, the electrons in the \( v = \frac{1}{3} \) FQH state have indeed condensed into a state with fractional charge excitations \( q^* = \frac{1}{3} e \). Many more fractions observed.

- Because \( \sigma_{xy} \to 0 \), this is a dissipationless state,
with a gap, presumably driven by the e\textsuperscript{−}-e\textsuperscript{−} Coulomb interaction.

- The new ideas developed to explore & understand these new phases of matter & their topological properties have had profound implications for physics, both in the lab, and the cosmos!

5.1 PRELIMINARIES: Mechanical Momentum + Guiding Centers

Recall

\[ H = \left( \frac{\vec{p} + e \vec{A}}{2m} \right)^2 = \frac{\vec{\Pi}^2}{2m} \]

where

\[ \vec{\Pi} = (\Pi_x, \Pi_y) = \vec{p} + e \vec{A}(\vec{r}) \]

This is a gauge-invariant quantity, but unlike the canonical momentum, \( \Pi_x \) & \( \Pi_y \) do not commute

\[ [\Pi_x, \Pi_y] = [p_x, eA_y] + [eA_x, p_y] = e\left( [\partial_x A_y, A_y] - [\partial_y A_x, A_x] \right) = -i\hbar e \left( \partial_x A_y - \partial_y A_x \right) = i\hbar e B = \frac{i\hbar^2}{e^2} \]

\[ B_z = -B \text{ (for convenience) \left( \frac{\hbar}{eB} = c \right)} \]