**Lecture 9**

**Detour: What is Berry**

**Phase(BP)**

BP describes phase accumulation due to a motion of some complex vector around a close loop in the complex vector space.

**Discrete Version**

For a specific example, let's consider a triatomic molecule.

The BP is defined as:

\[ \Phi = -\sum \ln \left[ \langle \psi_{1} \psi_{2} \psi_{3} \psi_{4} \rangle \right] \]

For a complex vector \( \mathbf{z} = 121 e^{i\theta} \), \( \text{Im} \ln \mathbf{z} = \Phi \).

Consider now our triatomic molecule:

\[
\begin{align*}
|u_{a}\rangle &= |u_{4}\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \\
|u_{b}\rangle &= \frac{1}{\sqrt{2}} \left( \begin{array}{c} e^{2\pi i/3} \\ 1 \end{array} \right) \\
|u_{c}\rangle &= \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -e^{i\pi/3} \end{array} \right)
\end{align*}
\]

Then BP is given by:

\[
\Phi = -\text{Im} \ln \left[ \langle u_{a} u_{b} u_{c} u_{4} \rangle \langle u_{4} u_{d} \rangle \right] = -\text{Im} \ln \left[ \left( \frac{e^{i\pi/3}}{2} \right)^{3} \right] = -\frac{3\pi}{2}.
\]

At least mathematically, the BP is independent of individual phases of \( |u_{i}\rangle \). Let's introduce a new set of \( N \) states:

\[
|0\rangle = \frac{1}{2^{N/2}} \sum_{j=0}^{2^{N-1}} e^{-i\beta j} |j\rangle
\]

We can show that in this case the BP is unaffected as \( e^{i\beta j} \) along the path will cancel out.

So we should say that **BP is gauge invariant** and as such perhaps describes some kind of physics.
THE CONCEPT OF A PARALLEL TRANSPORT

Suppose we have a chain of states \( |U_0 > \rightarrow |U_1 > \rightarrow \cdots \rightarrow |U_N > \) with no special phase relation. Let's define a new set, except of states \( \lambda U_0 > \lambda U_1 > \cdots \)

Continuous formulation of BP

In this formulation we parametrize the path by a real variable \( \lambda \) such that \( |U_\lambda > \) traverses the path as \( \lambda \) evolves from 0 to 1, i.e., \( |U_\lambda=0 > = |U_0 > \) and \( |U_\lambda > \) is a smooth function of \( \lambda \). Let's try to derive an expression similar to the discrete version.

\[
\text{Im} \left< U_{\lambda} | U_{\lambda} + d\lambda \right> = \text{Im} \left< U_{\lambda} | (1_{\lambda} + d\lambda \frac{d}{d\lambda}) + \cdots \right> \\
= \text{Im} \left< (1 + d\lambda \text{Re} \left< U_{\lambda} | d\lambda U_{\lambda} \right>) + \cdots \right> = d\lambda \text{Im} \left< U_{\lambda} | d\lambda U_{\lambda} \right> + \cdots
\]

Then BP is:

\[
\Phi = -\text{Im} \int d\lambda \left< U_{\lambda} | d\lambda U_{\lambda} \right>
\]

\[\text{Re} \left< U_{\lambda} | d\lambda U_{\lambda} \right> = \left< U_{\lambda} | d\lambda U_{\lambda} \right> + \left< d\lambda U_{\lambda} | U_{\lambda} \right> = \partial_{\lambda} \left< U_{\lambda} | U_{\lambda} \right> = 0
\]

\[\left< U_{\lambda} | d\lambda U_{\lambda} \right> \text{ is pure imaginary and}
\]

\[\Phi = \int d\lambda \left< U_{\lambda} | i d\lambda U_{\lambda} \right> \]

\[\text{Berry connection or Berry potential}
\]

\[\text{A}(\lambda) = \left< U_{\lambda} | i d\lambda U_{\lambda} \right> = -\text{Im} \left< U_{\lambda} | d\lambda U_{\lambda} \right>
\]

In terms of \( A(\lambda) \):

\[\Phi = \int d\lambda A(\lambda)
\]

Q: How Berry connection changes under gauge transformation?
\[ |\tilde{\psi}\rangle = e^{-i \beta (\lambda)} |\psi\rangle \uparrow \text{real function} \]

\[ A(\lambda) = \langle \tilde{\psi}_A | \lambda \rangle \tilde{\psi}_A \rangle = \langle \psi \lambda | e^{i \beta (\lambda)} \frac{\partial}{\partial \lambda} e^{-i \beta (\lambda)} |\psi \rangle = \langle \psi \lambda | \frac{\partial}{\partial \lambda} |\psi \rangle + \beta' (\lambda) \]

So BP is not gauge invariant! and it changes as:

\[ \tilde{A} = A + \frac{d \beta}{d \lambda} \] But what about BP?

\[ 1 \tilde{\psi}_{\lambda=1} = 1 \tilde{\psi}_{\lambda=0} \Rightarrow \beta_{\lambda=1} = \beta_{\lambda=0} + 2\pi \text{m} \]

\[ \int_0^1 \frac{d \beta}{d \lambda} d\lambda = \beta_{\lambda=1} - \beta_{\lambda=0} = 2\pi \text{m} \] so for

\[ \tilde{\phi} = \int \tilde{A}(\lambda) d\lambda = \int \left( A + \frac{d \beta}{d \lambda} \right) d\lambda = \phi + 2\pi \text{m} \]

So BP is still gauge invariant!

You can think of BP as the phase which still left over after moving in the loop.

**Example**

<table>
<thead>
<tr>
<th>[ \text{Let us illustrate this by considering a real physical problem.} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \text{Imagine we have an eigenvector which is a ground state of some } H \lambda. \text{ We can smoothly evolve the ground state by changing } \lambda, \text{ which in our case can be magnetic or electric fields } E. ]</td>
</tr>
<tr>
<td>[ \text{ same neutron at rest.} ]</td>
</tr>
<tr>
<td>[ H = -\gamma B \cdot \vec{s} = -\left( \frac{\gamma}{2} B \right) \hat{\omega} \cdot \vec{s} ]</td>
</tr>
<tr>
<td>[ \text{The ground state is independent of }</td>
</tr>
</tbody>
</table>
So we can write instead $|\nu_n\rangle$ to emphasize that $|\nu\rangle$ depends on the direction of the magnetic field and not on its magnitude $|B|$. 

Q: What's the BP of $|\nu_n\rangle$ as $\vec{B}$ carried around a loop in the magnetic field.

Let's try a simple discrete version:

1. $\vec{B} \rightarrow \text{rotate to } \vec{B}' \rightarrow \text{to } 2\vec{y} \rightarrow \text{back to } \vec{B}$.

So we are tracing one octant of the sphere.

\[ \phi = -\text{Im} \ln \left[ \left< \uparrow x \right| \left< \uparrow x \right| \right] \]

What we remember from QM 1 is that a spinor in arbitrary direction $\vec{B}$ is given by:

\[ |\uparrow \rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \ e^{i\varphi} \end{pmatrix} \]

\[ |\downarrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \]

Ignoring the normalization factors:

\[ \phi = -\text{Im} \ln \left[ (1)(1+i)(1) \right] = -\frac{\pi}{4} \]

Exercise:

Show that for $N$ spinors taking $N$ equally spaced values from 0 to $2\pi$ gives:

\[ \phi = -N \tan^{-1} \left[ \frac{\sin^2 \left( \frac{\pi}{2} \right) \sin \left( \frac{2\pi}{N} \right)}{\cos^2 \left( \frac{\pi}{2} \right) + \sin^2 \left( \frac{\pi}{2} \right) \cos^2 \left( \frac{2\pi}{N} \right)} \right] \]

a) $\phi$ for $N \to \infty$

b) find $\phi(\theta)$ for $N \to \infty$

c) for $\theta = 45^\circ$ compute numerically $N = 3, 4, 6, 12, \ldots$ and compare to $N \to \infty$

THE END