Phonons as excitations

THE PROBLEM OF QUANTUM MELTING

Second quantization:

Recall from quantum mechanics:

\[ [x, p] = xp - px = i \hbar \]

Based on this let's introduce 2 operators, which create and annihilate an excitation when applied to the ground state or vacuum.

\[
\begin{align*}
    b &= \frac{1}{\sqrt{2\hbar m\omega}} (Mwx + i p) \\
    b^+ &= \frac{1}{\sqrt{2\hbar m\omega}} (Mwx - i p)
\end{align*}
\]

Let's verify that anticommutator is:

\[
[b, b^+] = 1
\]

\[
[b b^+] = \frac{1}{2\hbar m\omega} \left\{ (M^2 \omega^2) [x x] + i M\omega [p x] - i M\omega [x p] + [p p] \right\} = \frac{1}{2\hbar} i M\omega [p x] = \frac{1}{2\hbar} (-2i [x p]) = \frac{-2i}{2\hbar} = 1
\]

Inversely we can express \( x \) and \( p \) as:

\[
\begin{align*}
    x &= \sqrt{\frac{\hbar}{2 m \omega}} (b^+ + b) \\
    p &= i \sqrt{\frac{2\hbar m\omega}{2}} (b^+ - b)
\end{align*}
\]

Recall these are operators here!

Just a different representation.

For the harmonic oscillator:

\[ H = \frac{p^2}{2M} + \frac{\beta x^2}{2} = \frac{\hbar^2}{2M} \omega^2 (b^+ b)^2 + \frac{\beta}{2M \hbar^2} (b^+ b)^2 = \frac{-\hbar^2}{4} (b^+ b)^2 + \frac{\beta}{2M \hbar^2} (b^+ b)^2 \leq B = \omega^2 M \]
\[ H = \frac{1}{2} \hbar \omega (b b^+ + b^+ b) = \hbar \omega \Sigma \]

\[ = \hbar \omega b^+ b + \frac{\hbar}{2} \hbar \omega = \frac{3}{2} \hbar \omega (n + \frac{1}{2}) \]

where \( n = b^+ b \) is the occupation number (the number of excitations in the ground state).

For the states with excitations present:

\[ b^i |n> = \sqrt{n} |n-i> \]

\[ b^i |n> = \sqrt{n+1} |n+1> \]

\[ \text{the philosophy of bosons: the more the merrier!} \]

\[ \text{the amplitude grows as } \sqrt{n+1}! \]

\( b \) - annihilation
\( b^+ \) - creation operators.

They increase (decrease) the \# of bosons by \( +1 \) (-1).

For \( n = b^+ b \) \( \Rightarrow \) \( h |n> = b^+ b |n> = b^+ \sqrt{n} |n-1> \)

\[ = \sqrt{n} \cdot \sqrt{n} |n> = n |n> \]

For the ground state: \( b |0> = 0 \)

\[ |n> = \frac{1}{\sqrt{n!}} (b^+)^n |0> \]

Now we are ready for something interesting.

\text{Quantum Melting.}