HW # 1 - Solutions

1. In the Einstein model, the atoms are treated as independent simple harmonic oscillators with a single frequency \( \omega_E \).

   By contrast in the Debye approach the atoms interact to produce collective lattice motions (e.g. sound waves) but there is assumed to be no interaction between these waves. As a result, a single wave does not decay or transform with time, and this model does not include thermal expansion.

2. In the Einstein model

\[
U(T) = \sum_n \left( n + \frac{1}{2} \right) \hbar \omega_E
\]

\( T = 0 \Rightarrow U = \frac{1}{2} \hbar \omega_E \) since \( n = 0 \)

For a harmonic oscillator \( \langle KE \rangle = \langle V \rangle \)

\[ \Downarrow \]

\[ \langle E \rangle = \langle KE \rangle + \langle V \rangle = \epsilon \langle V \rangle \]

\[ = 2 \left\{ \frac{1}{2} m \omega_E^2 \langle x^2 \rangle \right\} = \frac{1}{2} \hbar \omega_E \]

\[ \Downarrow \]

\[ \langle x^2 \rangle = \frac{\hbar}{2 m \omega_E} \]
Typically $w_E \sim 10^{13}$ s$^{-1}$

Use $m_p \sim 10^9$ eV/c$^2$

$$\langle x^2 \rangle = \frac{(hc) c}{4\pi (mc^2) (w_E)}$$

$$= \frac{(1.24 \text{ eV} \cdot \AA)}{(4\pi) (10^9 \text{ eV}) (10^{13} \text{ s}^{-1})}$$

$$\sim \frac{(3 \times 10^{18} \text{ } \AA} \text{ / s})}{(1.2) (3)} \frac{10^4}{10^{18}} = 0.29 \text{ } \AA^2$$

$$\sqrt{\langle x^2 \rangle} \sim 0.5 \text{ } \AA \text{ at } T = 0$$

3. For photons $w = kT$ (same as in Delye model with no cutoff)

$$U(T) = \int_0^\infty \frac{g(w) \hbar w \text{ dw}}{(e^{\hbar w/ kT} - 1)}$$

$$= \frac{\sqrt{\pi}}{\pi^{1/2}} \left( \frac{kT}{\hbar} \right)^4 \int_0^\infty \frac{x^3}{e^x - 1} \text{ dx}$$

$$\approx \frac{\pi^4}{15}$$
Therefore

\[ U(T) = \left( \frac{V}{\pi^2} \right) \left( \frac{\pi^4}{15} \right) \frac{(k_B T)^4}{(\hbar c)^3}. \]

\[ \downarrow \]

\[ C_V = \frac{\partial U}{\partial T} = 2k_B \left( \frac{V}{2\pi^2} \right) \left( \frac{\pi^4}{15} \right) \frac{(k_B T)^3}{(\hbar c)^3}. \]

\[ \downarrow \]

\[ \frac{C_V}{V} = \left( \frac{\pi^2}{15} \right) \left( \frac{k_B T}{\hbar c} \right)^3 \cdot k_B. \]

\[ \text{N.B.} \]

\[ \frac{C_V^{\text{lattice}}}{C_V^{\text{photon}}} \sim \left( \frac{C}{v} \right)^3 \sim \left( \frac{10^8 \text{ m/s}}{10^3 \text{ m/s}} \right)^3 \sim 10^{15}. \]

where \( v \) is the speed of sound.
\[ w = v_3 k^2 \implies k = \left( \frac{w}{v_3} \right)^{\frac{1}{2}}. \]

a) \( g(w) \)

Strategy:

(i) Calculate \( N(k) \)

(ii) Use \( w(k) \implies N(w) \)

(iii) \[ g(w) = 3 \frac{dN}{dw} \]

Polarization:

\[ N(k) = \frac{4}{3} \pi k^3 \frac{1}{(2\pi)^3 / V} = \frac{V}{6\pi^2} k^3 \]

where \( V = L^3 \)

\( \downarrow \)

\[ N(w) = \frac{V}{6\pi^2} \left( \frac{w}{v} \right)^{\frac{3}{2}} \]

\( \downarrow \)

\[ g(w) = 3 \frac{dN}{dw} = \frac{9}{2} \left( \frac{V}{6\pi^2 v} \right) \left( \frac{w}{v} \right)^{\frac{1}{2}}. \]
\[ g(w) = \left( \frac{3V}{4\pi^2v} \right) \left( \frac{w}{v} \right)^{1/2} \]

\[ 3N = \int_0^{w_{\text{max}}} g(w) \, dw \quad \text{defines} \quad w_{\text{max}}. \]

\[ 3N = \frac{3V}{4\pi^2v^{3/2}} \cdot \frac{w_{\text{max}}^{3/2}}{3/2} \]

\[ N = \left( \frac{V}{6\pi^2} \right) \frac{w_{\text{max}}^{3/2}}{v^{3/2}} \]

\[ w_{\text{max}} = \left( \frac{6\pi^2N}{V} \right)^{2/3} v \]
(c) \[ W(T) = \int_0^{\omega_{\text{max}}} \frac{g(\omega) \frac{\hbar \omega}{kT} d\omega}{\left( e^{\frac{\hbar \omega}{kT}} - 1 \right)} \]

\[ = \frac{V}{4\pi^2 v^{3/2}} \int_0^{\omega_{\text{max}}} \frac{\hbar \omega^{3/2} d\omega}{\left( e^{\frac{\hbar \omega}{kT}} - 1 \right)} \]

\[ e_V = \frac{dU}{dT} = \frac{V}{4\pi^2 v^{3/2}} \frac{\hbar^2}{kT^2} \int_0^{\omega_{\text{max}}} \frac{w^{5/2} e^{\frac{\hbar \omega}{kT}} d\omega}{\left( e^{\frac{\hbar \omega}{kT}} - 1 \right)} \]

below T behavior

\[ x = \frac{\hbar \omega}{k_b T}, \quad x_{\text{max}} \to \infty \]

\[ e_V = A \frac{1}{T^{7/2}} \int_0^{\infty} \frac{x^{5/2} e^x}{(e^x - 1)^2} dx \]

\[ e_V \propto T^{3/2} \Rightarrow \]

\[ e_V \propto T^{3/2} \]
5. 

\[ m \left\{ \frac{dv}{dt} + \frac{v}{r} \right\} = -eE \]

\[ \text{Let} \quad v = v_0 e^{-i\omega t} \]

\[ E = E_0 e^{-i\omega t} \]

\[ \downarrow \]

\[ (-i\omega + \frac{1}{r}) v_0 = -eE_0 / m \]

\[ \downarrow \]

\[ v_0 = -\frac{eE_0}{m} \quad = -\frac{eE}{m} \quad \frac{1 + i\omega \tau}{1 + (\tau \omega)^2} \]

\[ j = n(-e)v = \frac{ne^2}{m} \quad \frac{1 + i\omega \tau}{1 + (\tau \omega)^2} \quad E \]

\[ = \sigma E \]

\[ \downarrow \]

\[ \sigma(\omega) = \sigma(0) \quad \frac{1 + i\omega \tau}{1 + (\tau \omega)^2} \]
\[ \rho \sim \frac{m}{N \sigma^2} \quad \tau = \frac{1}{\sigma^2} \]

Random thermal \implies equipartition theorem

\[ \frac{1}{2} m \langle v^2 \rangle \sim \frac{1}{2} m \omega^2 \langle x^2 \rangle \sim T \]

\[ l \sim n = 1 \implies l \sim \frac{1}{\sigma} \]

\[ l \sim \frac{1}{\sigma} \sim \frac{1}{\pi \langle x^2 \rangle} \sim \frac{1}{T} \]

\[ v \sim T^{1/2} \]

\[ \rho \sim \frac{1}{l^3} = \frac{1}{T^{3/2}} \implies \rho \sim T^{3/2} \]
7. a) The probability that an electron does not suffer a collision in \( \frac{dt}{t} \) is

\[
\lim_{dt \to 0} \left(1 - \frac{dt}{t}\right)^{t/dt} = \lim_{dt \to 0} \left\{ \left(1 + \left(-\frac{dt}{t}\right)\right)^{-\frac{t}{dt}} \right\}^{\frac{-t}{t}} = e
\]

where we have used

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e
\]

Symmetry in time \( \Rightarrow \) backwards, forwards same solution

Therefore, the probability of an electron picked at random that has not had a collision in the preceding \( t \) seconds is \( e^{-t/\tau} \).
1. Probability = \( \text{(Probability of having no collision in time } t) \times \text{(probability of having a collision in } dt) \)
   \[ = e^{-t/\tau} \int_0^t dt \]

3) Mean time back to the last collision averaged over all the electrons.

\[ t_0 = \int_0^t \frac{t \cdot dne(t)}{\int_0^t dne(t)} = \left( \# \text{ electrons that have not scattered in time } t \right) \times t \]

where \( n_e(t) \) is the \# of electrons that have not had a collision in time \( t \)

\[ n_e(t) \propto e^{-t/\tau} \]

\[ N = A \int e^{-t/\tau} dt = \Lambda \]

\[ A = N/\gamma \]

\[ dne(t) = \frac{N}{\tau} e^{-t/\tau} dt \]
\[
\bar{t}_c = \frac{N}{T} \int_0^\infty t e^{-t/\tau} \, dt
\]

\[
= \frac{T^2}{\tau} = \tau.
\]

\[
\frac{N}{T} \int_0^\infty e^{-t/\tau} \, dt
\]

---

**d) Mean time between successive collisions of a single electron**

\[
\bar{t}_d = (\text{Probability of no collisions in time } t)
\]

\[\times (\text{Probability of a collision in time } dt) \times t
\]

\[
\bar{t}_d = \frac{N}{T} \int_0^\infty t e^{-t/\tau} \, dt
\]

\[
= \frac{T^2}{\tau} = \tau
\]

(time average)
Internal between collisions = \( T \) averaged over all electrons.

\[
T = t_1 + t_2
\]

\[
t_2 = T - t_1
\]

\( \bar{t}_e \) = time between next and last collision averaged over all electrons.

Using our results from c)

\[
P_e(t = T) = \int_0^T \int_0^{t_2/\tau} e^{-t_1/\tau} e^{-t_2/\tau} \delta(t_1 + t_2 - \tau) dt_1 dt_2
\]

\[
= \int_0^T e^{-t_1/\tau} e^{-(T-t_1)/\tau} e^{-T/\tau} dt_1 = \frac{T}{\tau^2} e^{-T/\tau}
\]

\( \bar{t}_e \) = mean time between last + next collision averaged over all e's.

\[
= \int_0^\infty \int_0^{T/\tau} \frac{T^2}{\tau^2} e^{-T/\tau} dT / \int_0^\infty \int_0^{T/\tau} \frac{T}{\tau^2} e^{-T/\tau} dT
\]

\[
c = \frac{T}{\tau} = \tau \int_0^\infty x^2 e^{-x} dx / \int_0^\infty e^{-x} dx
\[ t_e = \tau \int_0^\infty \frac{x^2}{e^{-x}} \, dx = 2 \tau \]

\[ \frac{2}{\int_0^\infty \frac{x^2}{e^{-x}} \, dx} = 1 \]

\( t_d \) is the mean time between successive collisions of a single electron \((= \tau)\).

\( t_e \) is the time between the last and the next collision averaged over all electrons.

In the Drake equation, we need a time that is the inverse of the scattering rate of a single electron. This is the timescale associated with current decay.