1) The Lagrangian of a system is given by \( L = \frac{1}{2} \alpha x^2 \), with \( \alpha > 0 \). For the initial condition \( x(0) = 0 \), \( \dot{x}(0) = v \), solve the equation of motion to obtain \( x(t) \) for \( t > 0 \). [10]

2) On an immobile planet with spherically symmetric mass distribution, someone is shooting a cannonball trying to hit the antipode. The shooter points at an angle \( \gamma \) from the vertical direction with a speed \( u \). The mass of the planet is \( M \) and the radius is \( R \). The cannonball has mass \( m \) and the gravitational attraction force \( F = \frac{GMm}{r^2} \) at a distance \( r \) from the center of the planet.
a) Find the condition satisfied by $u$ and $r$.

b) The shooter wants to use the least initial kinetic energy. What angle $\gamma$ would he shoot at?

3) A gyroscope, supported at one end, is described by the Lagrangian in terms of Euler angles:

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta$$

The gyroscope is held at $\phi = \phi_0$, $\theta = \frac{\pi}{2}$, $\dot{\phi} = 0$, $\dot{\theta} = 0$, $\dot{\psi} = \omega_3$ at $t = 0$, and then let $g = 0$. The gyroscope starts falling and turning under gravity.

Assume \( \left( \frac{I_3}{I_1} \right) \frac{I_3 \omega_3^2}{2Mgl} \gg 1 \).
a) How far does it fall before turning up, to leading order? [10]

b) Imagine gravity is suddenly ‘turned off’ at the lowest point. Describe the motion from that point on. [20]