Final solutions (2021)

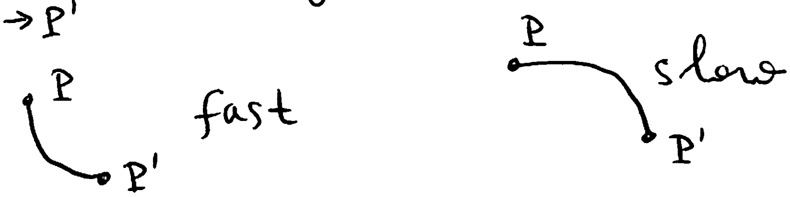
(1.) (a) Use energy conservation:

$$T(P) = 0, \quad T(P') = \frac{m v(P')^2}{2}.$$

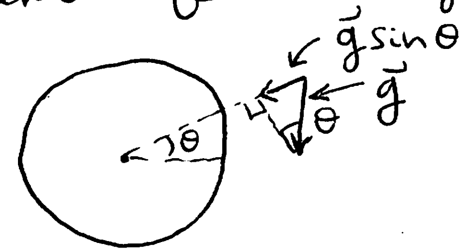
$$\frac{\cancel{m} v(P')^2}{2} = \frac{\cancel{m} g H}{3} \quad \text{gives}$$

$$v(P') = \sqrt{\tfrac{2}{3} g H}.$$

$v(P')$ will be the same for all ramp shapes. However, we expect $t_{P \to P'}$ to depend on the ramp shape:



(b) m will leave the surface once the centrifugal force is equal to the normal component of the gravitational force:



$$\frac{\cancel{m} v^2}{H} = \cancel{m} g \sin\theta \quad (*)$$

We need to find v at the point of departure : $\frac{1}{2}v^2 = \underbrace{\frac{gH}{3}}_{\text{from } v(P')} + g(H - H\sin\theta)$.

$$\text{But} \quad H\sin\theta \overset{(*)}{=} \frac{v^2}{g}, \text{ yielding}$$

$$\frac{v^2}{2} = \frac{4}{3}gH - v^2,$$

$$v^2 = \frac{8}{9}gH.$$

departure velocity

The height (above ground) at which it loses contact w/the slide is: $H + H\sin\theta = H + \frac{8}{9}H = \frac{17}{9}H$.

2.

(a) EoM:

$$\begin{cases} m_1 \ddot{x}_1 = -k_1 x_1 - k_{12}(x_1 - x_2), \\ m_2 \ddot{x}_2 = -k_{12}(x_2 - x_1) - k_2 x_2 \end{cases}$$

(b) Use $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} e^{i\omega t}$:

$$\begin{cases} (-m_1\omega^2 + k_1 + k_{12})\, x_{1,0} - k_{12}\, x_{2,0} = 0, \\ -k_{12}\, x_{1,0} + (-m_2\omega^2 + k_2 + k_{12})\, x_{2,0} = 0. \end{cases}$$

Matrix form:

$$\begin{pmatrix} \frac{k_1 + k_{12}}{m_1} - \omega^2 & -\frac{k_{12}}{m_1} \\ -\frac{k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2} - \omega^2 \end{pmatrix} \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} = 0 .$$

Plug in numbers:

$$\begin{pmatrix} 8 - \omega^2 & -4 \\ -4 & 8 - \omega^2 \end{pmatrix} \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} = 0 .$$

$$\det = 0 \Rightarrow (\omega^2 - 8)^2 - 16 = 0,$$

$$\omega^2 = 8 \pm 4 = \begin{cases} 12 \\ 4 \end{cases}$$

So, $\begin{cases} \omega_1 = \sqrt{12}, \\ \omega_2 = 2 \end{cases} \left(\frac{rad}{s}\right)$

ω_1: $-4(x_{1,0} + x_{2,0}) = 0 \Rightarrow x_{1,0} = -x_{2,0}$

ω_2: $4(x_{1,0} - x_{2,0}) = 0 \Rightarrow x_{1,0} = x_{2,0}$

(c) Use

$$\begin{cases} x_1(t) = C_1 \cos(\omega_1 t) + C_2 \cos(\omega_2 t), \\ x_2(t) = -C_1 \cos(\omega_1 t) + C_2 \cos(\omega_2 t) \end{cases}$$

no initial phases since $\dot{x}_1(0) = \dot{x}_2(0) = 0$.

$x_1(0) = 0 \Rightarrow C_1 = -C_2$.

$x_2(0) = 0.1 \Rightarrow 2C_2 = 0.1$, $C_2 = 0.05$ m.

Finally,

$$\begin{cases} x_1(t) = -0.05 \cos(\sqrt{12}\, t) + 0.05 \cos(2t), \\ x_2(t) = 0.05 \cos(\sqrt{12}\, t) + 0.05 \cos(2t). \end{cases}$$

(3.) (a) Mass of liquid in the tube: $\rho a h$. Kinetic energy of the liquid:

$T = \frac{1}{2}(\rho a h)\dot{h}^2$. Potential energy wrt the reservoir: $V = \frac{1}{2} g \rho a (h-H)^2$.

Then $\mathcal{L} = T - V = \frac{\rho a h}{2}\dot{h}^2 - \frac{g\rho a}{2}(h-H)^2$.

next,

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{h}} = \rho a h \dot{h}, \quad \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{h}}\right) = \rho a \dot{h}^2 + \rho a h \ddot{h}. \\ \dfrac{\partial \mathcal{L}}{\partial h} = \dfrac{\rho a}{2}\dot{h}^2 - g\rho a (h-H). \end{cases}$$

EoM: $h\ddot{h} + \dot{h}^2 - \frac{\dot{h}^2}{2} + g(h-H) = 0$, or

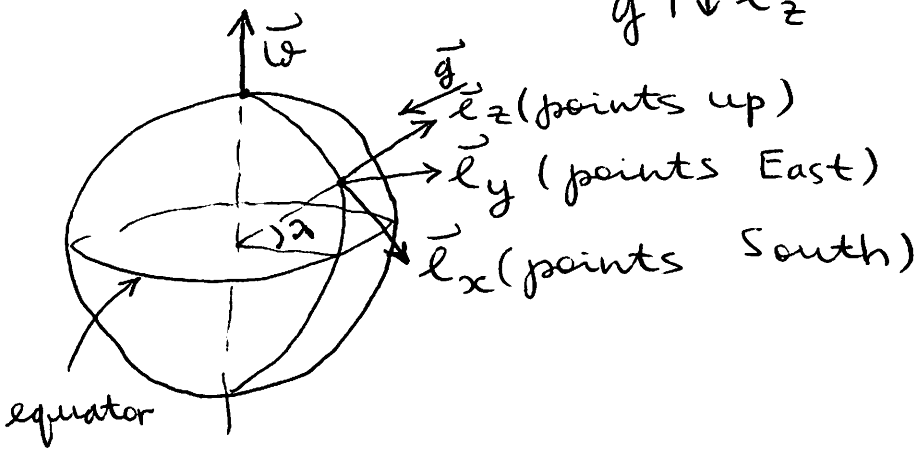
$$h\ddot{h} + \frac{\dot{h}^2}{2} + g(h-H) = 0. \qquad (*)$$

(b) Eq (*) is different from the SHO eq'n: $\ddot{h} + \alpha(h-H) = 0$ due to the h factor in the acceler'n term and the $\frac{\dot{h}^2}{2}$ term. This is basically because the mass in the tube is not constant, and because the system is open (liquid flows out into the reservoir).

(4.) $\vec{F}_{eff} = m(\underbrace{\vec{g} - 2\vec{\omega}\times\vec{v}}_{\vec{a}})$.

Coordinate system:

$$\vec{g} \uparrow\downarrow \vec{e}_z$$



$$\begin{cases} \omega_x = -\omega\cos\lambda, \\ \omega_y = 0, \\ \omega_z = \omega\sin\lambda. \end{cases}$$

To $O(\omega)$, we can neglect higher-order Coriolis effects and take

$$\begin{cases} \dot{x} \simeq 0, \\ \dot{y} = 0, \\ \dot{z} \simeq -gt \end{cases}$$

[all corrections would be $O(\omega^2)$]

Then $\vec{\omega}\times\vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & 0 & -gt \end{vmatrix} =$

$$= -\omega g t\cos\lambda\,\vec{e}_y$$

Since $\vec{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$,

$$\vec{a} = \begin{pmatrix} 0 \\ 2\omega g t \cos\lambda \\ -g \end{pmatrix}$$ ← Coriolis effect, force in the easterly direction

So, $\ddot{y} \overset{O(\omega)}{\simeq} 2\omega g t \cos\lambda$.

$y(0) = \dot{y}(0) = 0$:

$$y(t) = \frac{1}{3}\omega g t^3 \cos\lambda$$

Time of fall: $t \simeq \sqrt{\frac{2h}{g}}$.

Finally,

$$d \simeq \frac{1}{3}\omega\cos\lambda\left(\frac{2h}{g}\right)^{3/2} g = \frac{\omega\cos\lambda}{3}\left(\frac{8h^3}{g}\right)^{1/2}.$$

(b) For $h = 10^2$ m and $\lambda = \frac{\pi}{4}$, we obtain: $d \simeq 1.55$ cm.

5. Recall that

$$\mathcal{L}_z = x p_y - y p_x . \qquad \left(\mathcal{J}_z \equiv L_z \right)$$

Then
$$[u, \mathcal{L}_z] = \frac{\partial u}{\partial x}\frac{\partial \mathcal{J}_z}{\partial p_x} + \frac{\partial u}{\partial y}\frac{\partial \mathcal{J}_z}{\partial p_y} +$$

$$+ \frac{\partial u}{\partial z}\underbrace{\cancel{\frac{\partial \mathcal{J}_z}{\partial p_z}}}_{0} - \frac{\partial u}{\partial p_x}\frac{\partial \mathcal{J}_z}{\partial x} - \frac{\partial u}{\partial p_y}\frac{\partial \mathcal{J}_z}{\partial y} -$$

$$- \frac{\partial u}{\partial p_z}\underbrace{\cancel{\frac{\partial \mathcal{J}_z}{\partial z}}}_{0} = \frac{\partial u}{\partial x}(-y) + \frac{\partial u}{\partial y} x -$$

$$- \frac{\partial u}{\partial p_x} p_y - \frac{\partial u}{\partial p_y}(-p_x) .$$

Now,
$$\frac{\partial u}{\partial x} = \overbrace{\frac{\partial u}{\partial r^2}}^{u'}\frac{d r^2}{dx} + \overbrace{\frac{\partial u}{\partial (\vec{r}\cdot\vec{p})}}^{\tilde{u}'}\frac{d(\vec{r}\cdot\vec{p})}{dx} =$$

$$= u' \times (2x) + \tilde{u}' p_x \text{, etc.}$$

Then
$$[u, \mathcal{L}_z] = (-2x\cancel{y})\, u' - y p_x \tilde{u}' +$$

$$+ (2x\cancel{y})\, u' + p_y x \tilde{u}' - [u'\cancel{\times(2p_x)} + \tilde{u}' x] p_y +$$

$$+ [u'\cancel{\times(2p_y)} + \tilde{u}' y] p_x = \tilde{u}'[-y p_x + p_y x - x p_y +$$

$$+ y p_x] = \underline{\underline{0}}, \text{ as desired.}$$