13.2 Equal and Opposite Magnetization

(a) There is no free current. The magnetization in each region is uniform so the bulk magnetization current density \( j_M = \nabla \times M = 0 \). The magnetization is normal to the \( z = 0 \) interface so the surface magnetization current density \( K = M \times \hat{n} = 0 \). There is no source current of any kind, so \( B = 0 \) everywhere.

(b) There is no bulk magnetic charge \( \rho^* = -\nabla \cdot M \) but there is a surface charge density \( \sigma^* = M \cdot \hat{n} \). There is a contribution \( \sigma = M \) at \( z = 0 \) due to the \( z > 0 \) region. An identical contribution comes from the \( z < 0 \) region. Therefore, since an outward-pointing electric field \( E = \sigma/2\varepsilon_0 \) is created by a planar surface density of electric charge \( \sigma \), we get an outward-pointing field \( H = M \) in this case. Since \( M \) points inward to the same interface, we conclude that \( B = \mu_0(H + M) = 0 \) everywhere.

13.5 The Virtues of Magnetic Charge

(a) The text establishes that \( m = \int d^3r M \). On the other hand, using the proposed formula, the \( k^{\text{th}} \) component of the magnetic dipole moment of the sample is

\[
m_k = -\int d^3r r_k \nabla \cdot M = -\int d^3r \nabla \cdot (M r_k) + \int d^3r (M \cdot \nabla) r_k = \int d^3r M_k.
\]
(b) By definition, the interaction energy between two current distributions is

\[ \hat{V}_B = -\frac{\mu_0}{4\pi} \int d^3r \int d^3r' \frac{j_1(r) \cdot j_2(r')}{|r - r'|}. \]

Using the definition of the vector potential in the Coulomb gauge, this is

\[ \hat{V}_B = -\int d^3r j_1 \cdot A_2 = -\int d^3r A_2 \cdot \nabla \times M_1 = \int d^3r \nabla \cdot (A_2 \times M_1) - \int d^3r M_1 \cdot \nabla \times A_2. \]

Finally, using the divergence theorem and the fact that \( M_1 \) is zero on the integration surface at infinity, we conclude that

\[ \hat{V}_B = -\int d^3r M_1 \cdot B_2. \]

Precisely the same steps beginning with \( \hat{V}_B = -\int d^3r j_2 \cdot A_1 \) establish the reciprocity relation.

(c) It is simplest to begin with the proposed formula and show that it is equivalent to the expression derived in part (b). Then, because \( B_2 = \mu_0 H_2 \) in the part of space where \( M_1 \neq 0 \),

\[ \hat{V}_B = \frac{\mu_0}{4\pi} \int d^3r \int d^3r' \frac{\nabla \cdot M_1(r) \nabla' \cdot M_2(r')}{|r - r'|} = \frac{\mu_0}{4\pi} \int d^3r' \nabla' \cdot M_2(r') \int d^3r \left\{ \nabla \cdot \left[ \frac{M_1(r)}{|r - r'|} \right] - M_1(r) \cdot \nabla \frac{1}{|r - r'|} \right\} = \int d^3r M_1(r) \cdot \nabla \frac{\mu_0}{4\pi} \int d^3r' \frac{\nabla' \cdot M_2(r')}{|r - r'|} = -\int d^3r M_1(r) \cdot \mu_0 H_2(r) = -\int d^3r M_1(r) \cdot B_2(r). \]
13.11 Lunar Magnetism

We have $B = \mu_0 (H + M)$, where $H = -\nabla \psi$ and $\psi$ satisfies the Poisson-like equation

$$\nabla^2 \psi = \nabla \cdot M.$$ 

In addition, at the boundary between regions, it is necessary to satisfy the matching conditions

$$\psi_1(r_S) = \psi_2(r_S)$$

and

$$\left[ \frac{\partial \psi_1}{\partial n_1} - \frac{\partial \psi_2}{\partial n_1} \right] = [M_1 - M_2] \cdot \hat{n}_1.$$

We call the core, crust, and exterior of the Moon regions I, II, and III, respectively, as shown below.

The impressed magnetization $M$ of the core is stated to be proportional to a dipole field $B_d$ centered at the origin. If we align the magnetic moment $m$ with the $z$-axis,

$$B_d(r, \theta) = \frac{\mu_0 m}{4\pi} \frac{3 \cos \theta \vec{r} - \vec{z}}{r^3} = \frac{\mu_0 m}{4\pi} \frac{2 \cos \theta \vec{r} + \sin \theta \hat{\theta}}{r^3}.$$

Since $\nabla \cdot B = 0$, we know that $\nabla \cdot M = 0$ and the magnetic scalar potential above satisfies Laplace's equation everywhere. Specifically,

$$\psi_I = D \left( \frac{r}{b} \right) \cos \theta$$

$$\psi_{II} = \left[ B \left( \frac{a}{r} \right)^2 + C \left( \frac{r}{a} \right) \right] \cos \theta$$

$$\psi_{III} = A \left( \frac{a}{r} \right)^2 \cos \theta.$$

Applying the matching conditions, noting that $\hat{n} = \hat{r}$ and that the only non-zero magnetization is

$$M_{II} = M \frac{2 \cos \theta \vec{r} + \sin \theta \hat{\theta}}{r^3},$$

gives

$$A = B + C$$

$$D = B \frac{a^2}{b^2} + C \frac{b}{a}$$

$$\frac{2M}{a^3} = -\frac{2B}{a} + \frac{C}{a} + \frac{2A}{a}$$

$$\frac{2M}{b^3} = \frac{D}{b} + 2B \frac{a^2}{b^3} - \frac{C}{a}.$$

It is straightforward to check that this system is solved by

$$A = 0 \quad C = -B = \frac{2M}{3a^2} \quad D = B \left[ \frac{a^2}{b^2} - \frac{b}{a} \right],$$

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which confirms that $\mathbf{H} = \mathbf{B} = 0$ in region III outside the Moon. We can sketch $\mathbf{B}$ inside the Moon using the fact that $\mathbf{B}_l = \mathbf{H}_l = -\nabla \psi_l$ is constant, the lines of $\mathbf{B}$ must form closed loops, and $\mathbf{B}$ must be tangent to the sphere at $r = b$ because its radial component is continuous there.