Lab 8: The onset of chaos

Unpredictable dynamics of deterministic systems, often found in non-linear systems.

- Background of chaos
- Logistic Equation
- Non-linear RLC circuit.

What is Chaos?

In common usage, “chaos” means random, i.e. disorder. However, this term has a precise definition that distinguish chaos from random.

Chaos $\neq$ Random

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

-Edwards Lorenz

Chaos theory

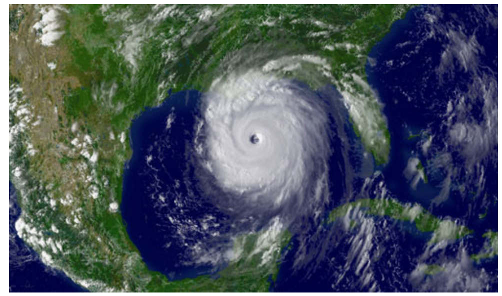
Chaos theory studies the behavior of dynamical systems that are **highly sensitive to initial conditions**, an effect which is popularly referred to as the *butterfly effect*.

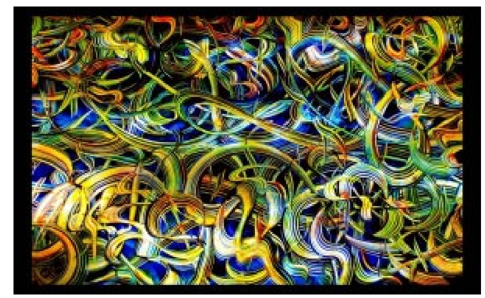
Edward Norton Lorenz

Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas? - Edward Lorentz, AAAS, (1972)

Applications of Chaos theory

- meteorology
- sociology
- physics
- engineering
- aerodynamics
- economics
- biology
- philosophy.





Is there any universal behavior in chaos phenomena?

Logistic equation

Logistic equation has been used to model population swing of organisms, e.g. lemmings.

$$x(n+1) = a \cdot x(n) \cdot \left[1 - x(n)\right]$$

$x(n)$: n-th year's population;

a: growth parameter, i.e. birth/death;

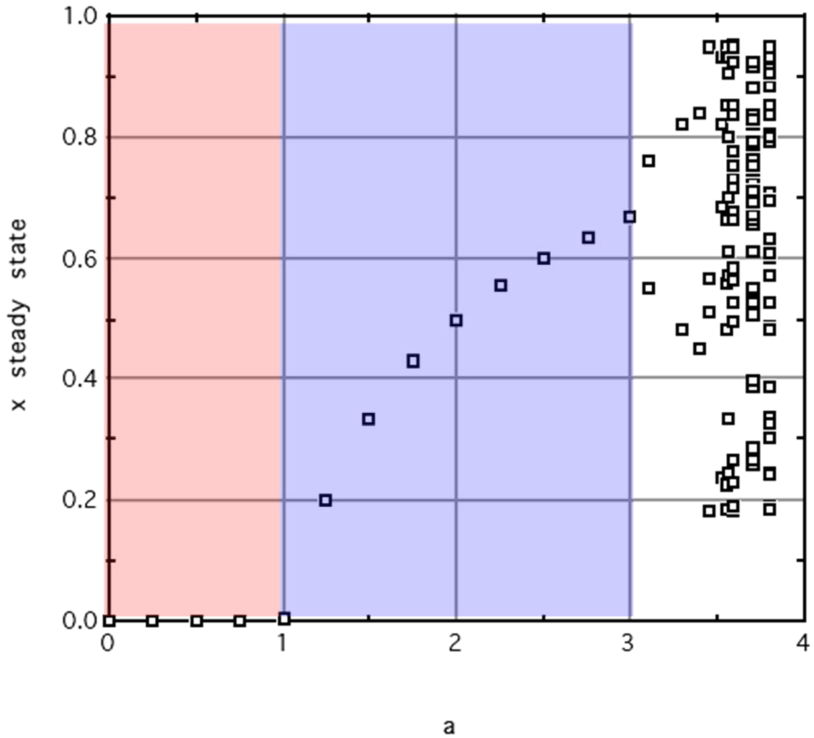
$[1 - x(n)]$: competition within population.

The grand challenge: what is the steady state of the organism's population in the long run ($n \rightarrow \infty$)? How does it depends on growth parameter a?

Run LabView program

Logistic equation

It depends on the growth parameter *a*.



$$0 \le a < 1 \quad x(\infty) = 0$$

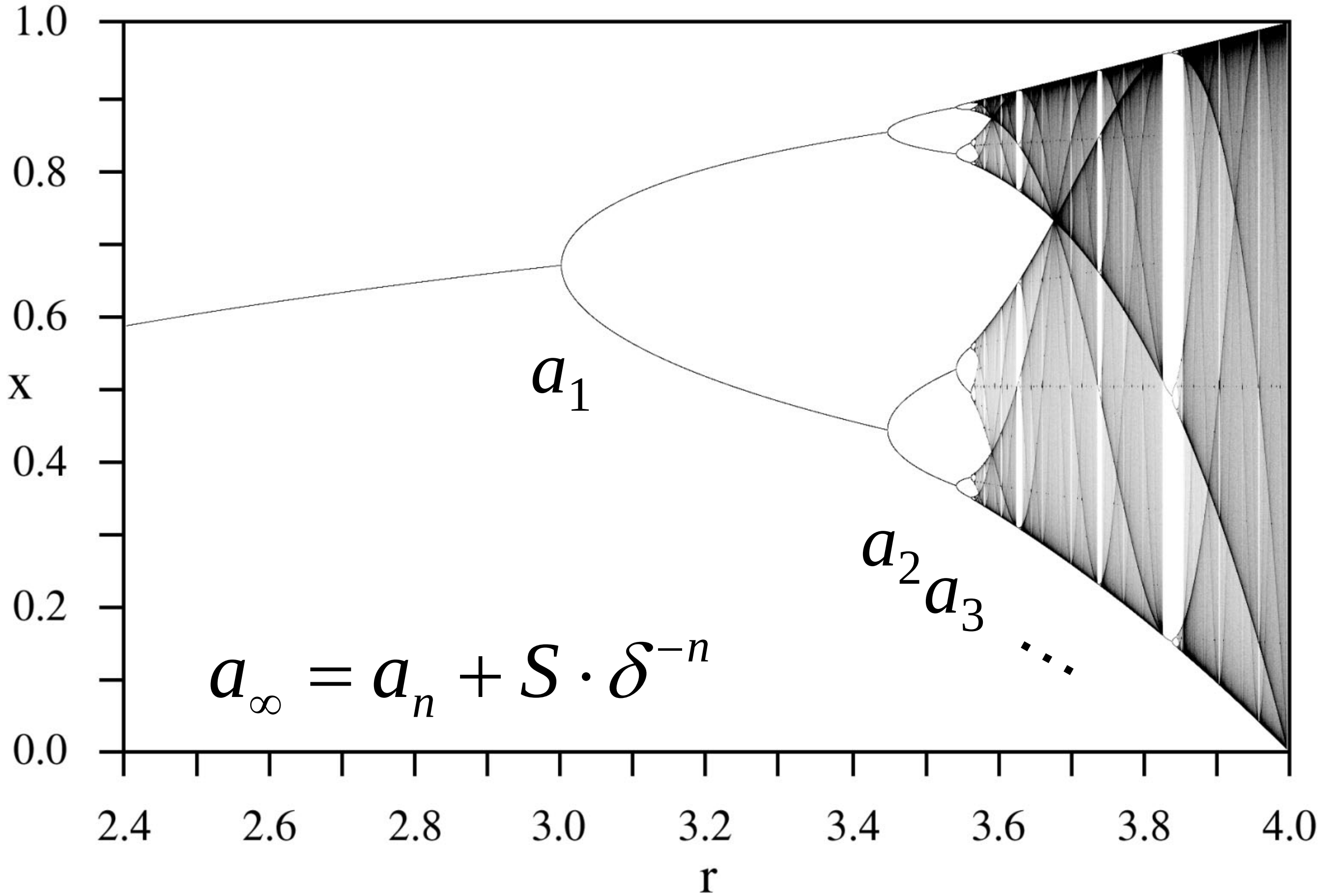
$$1 \le a < 3 \quad x(\infty) = 1 - 1/a$$

$3 \le a < 3.4495$ period doubles

$a \ge 3.6$ chaos set in

Is there any universal behavior?

Logistic equation and onset of chaos



Feigenbaum number: $\delta = 4.6692...$

Mitchell Feigenbaum (July 1978) "Quantitative universality for a class of nonlinear transformations," *Journal of Statistical Physics*, vol. 19, no. 1, pages 25–52.

How to find out δ and S from measurements?

$$a_\infty = a_n + S\cdot\delta^{-n} \qquad a_\infty = a_{n-1} + S\cdot\delta^{-(n-1)}$$

$$\Rightarrow a_n - a_{n-1} = S\cdot(\delta-1)\cdot\delta^{-n}$$

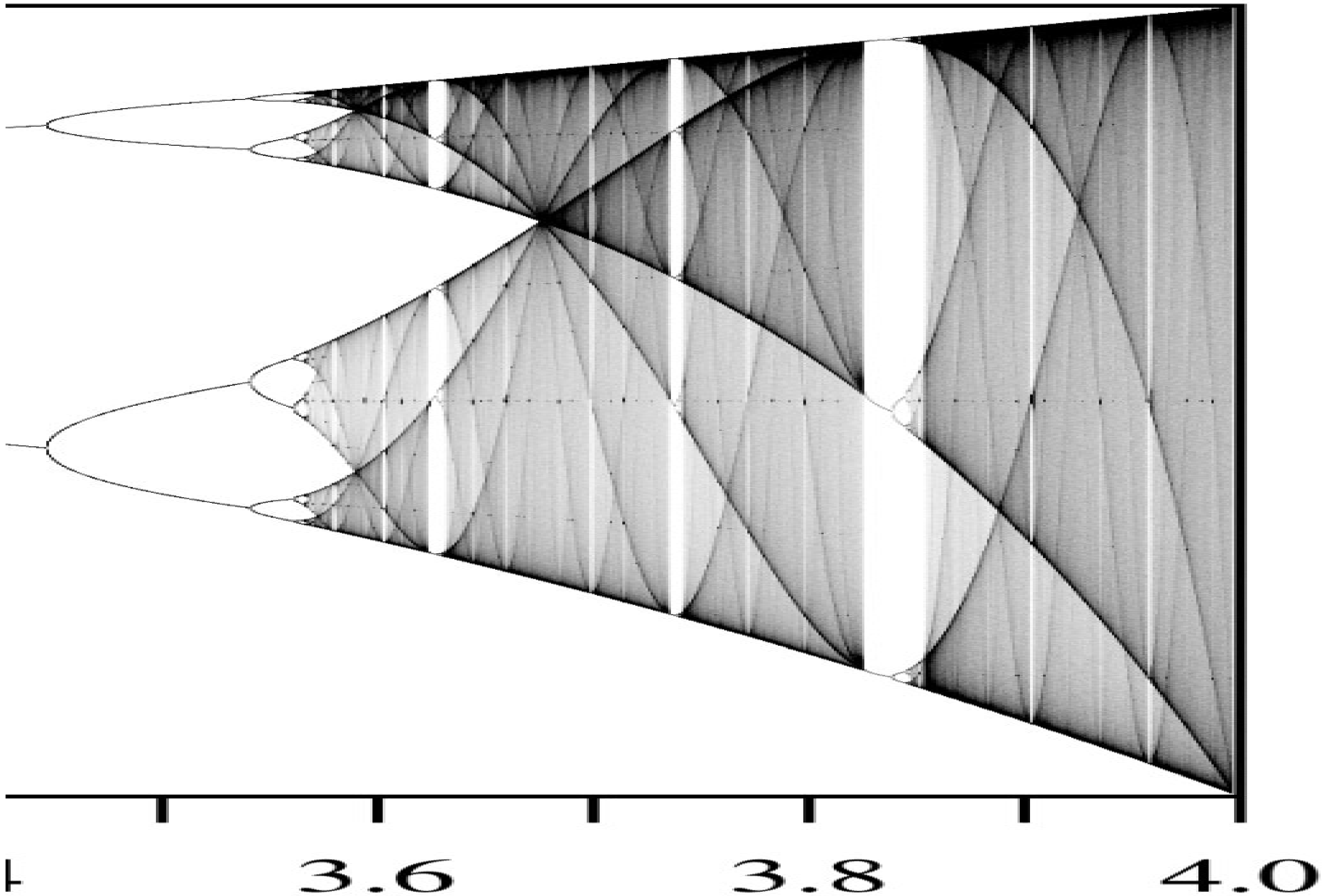
$$\Rightarrow\ a_{n+1} - a_n = S\cdot(\delta-1)\cdot\delta^{-n-1} = S\cdot(\delta-1)\cdot\delta^{-n}/\delta$$

$$\Rightarrow \delta = \frac{a_n - a_{n-1}}{a_{n+1} - a_n}$$

$$S = \frac{\delta^n\cdot(a_n - a_{n-1})}{\delta - 1}$$

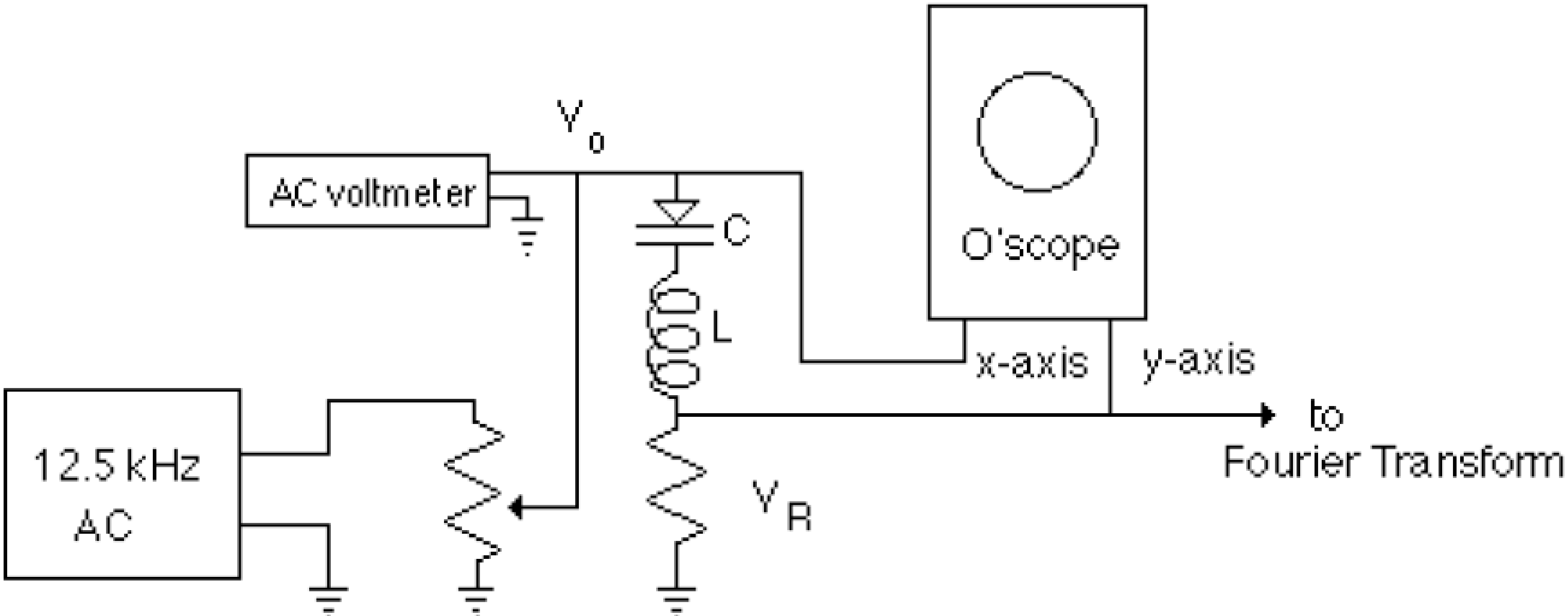
Feigenbaum number is a universal number for non-linear dynamical systems

Complex but organized structure inside chaos



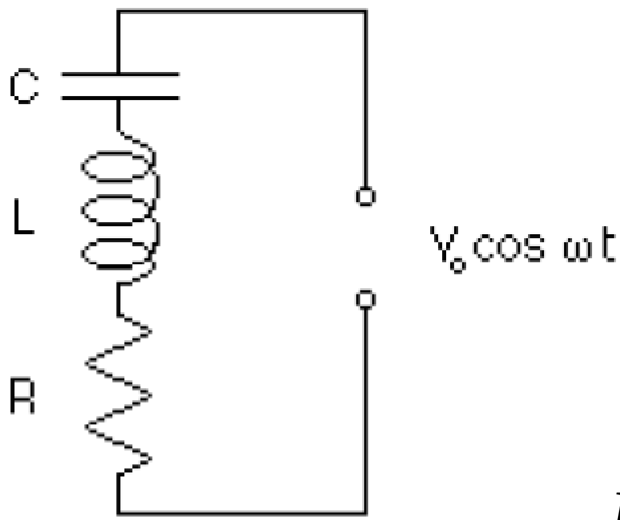
Another universal feature; Emergent complexity

Another example: non-linear RLC circuit



$$C(V) = \frac{C_0}{\left(1 + V/\varphi\right)^{\gamma}}$$

Linear RLC circuit ⇔ harmonic motion



$$q$$

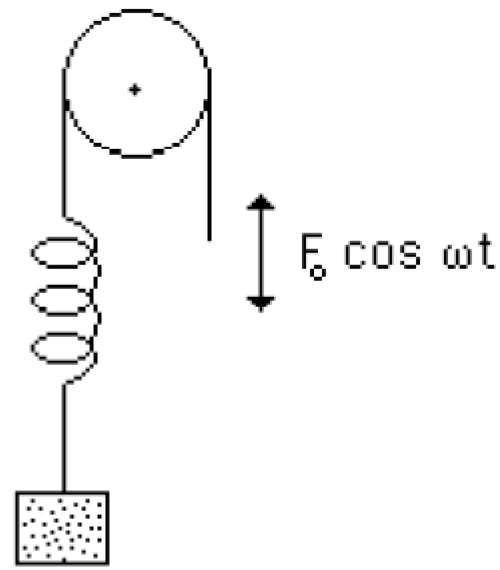
$$I = \dot{q}$$

$$\ddot{q}$$

$$V = \tfrac{1}{C} q$$

$$L\dot{I} = -V$$

$$V = IR$$



$$x$$

$$v = \dot{x}$$

$$a = \ddot{x}$$

$$F = -kx$$

$$F = ma$$

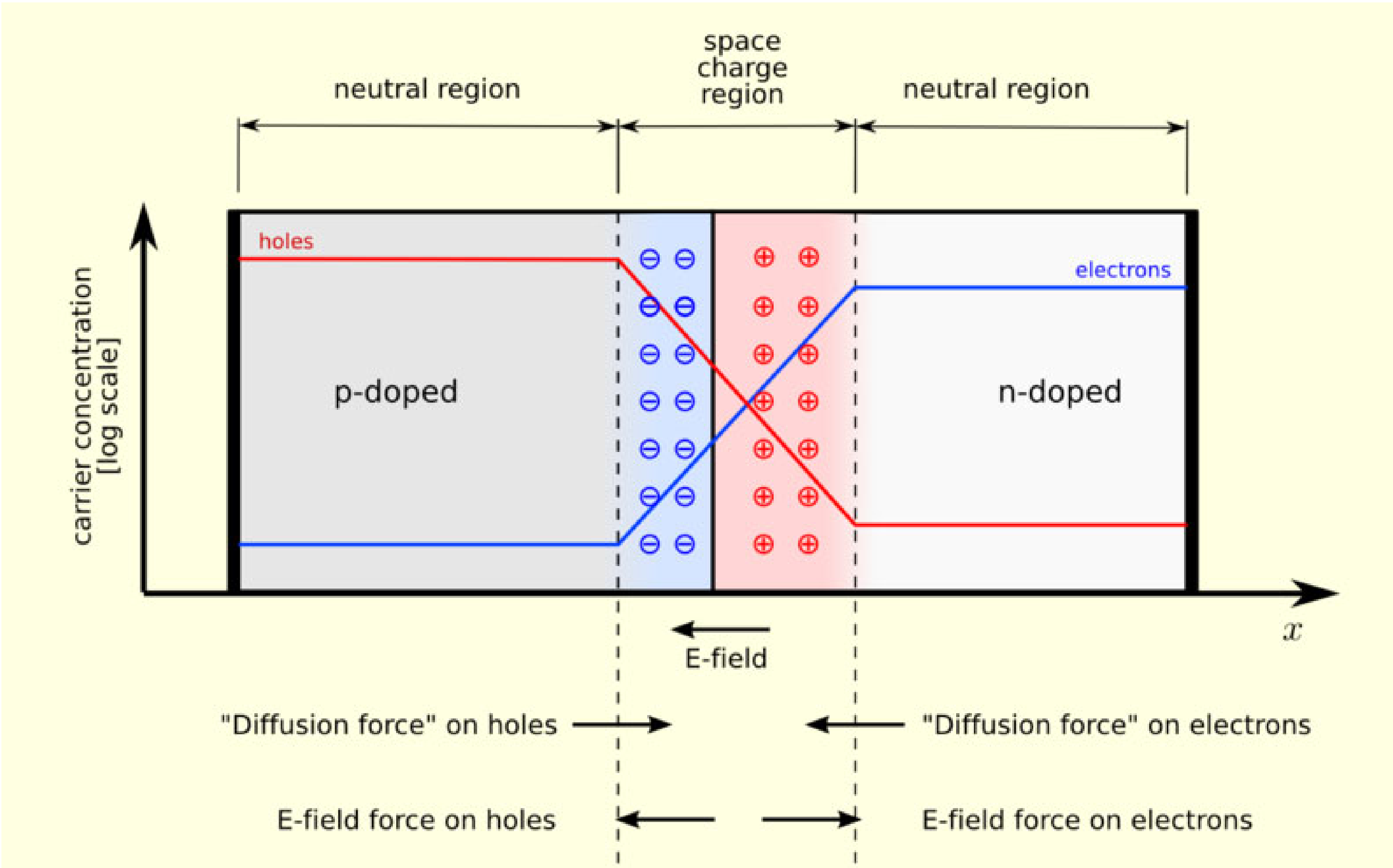
$$F_{damp} = -Rv$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC} q = V_0 \cos \omega t$$

$$\ddot{x} + \frac{R}{m}\dot{x} + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t$$

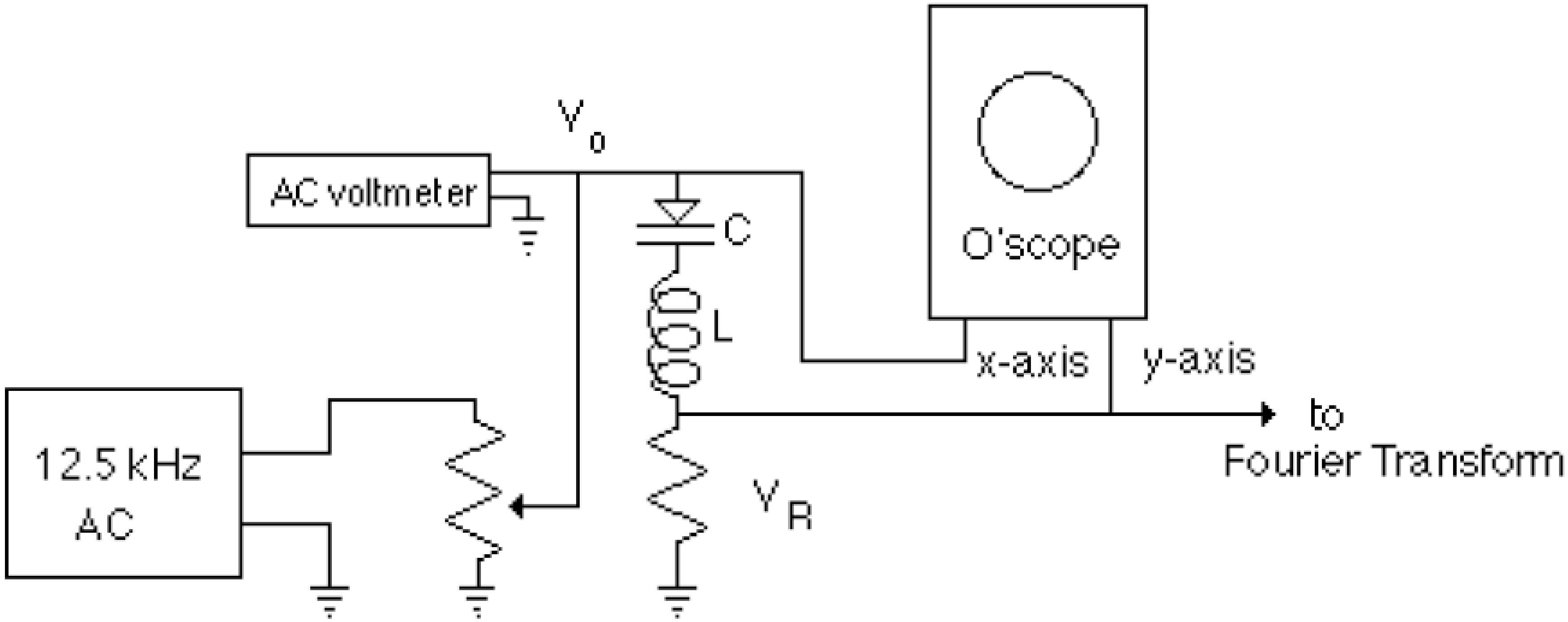
Non-linear RLC circuit ⇔ aharmonic motion

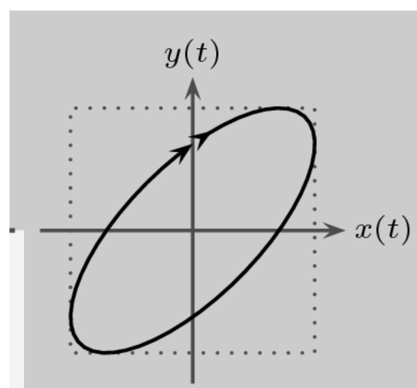
Nonlinear capacitor: p-n junction



$$C(V)=\frac{C_0}{(1+V/\varphi)^{\gamma}}$$

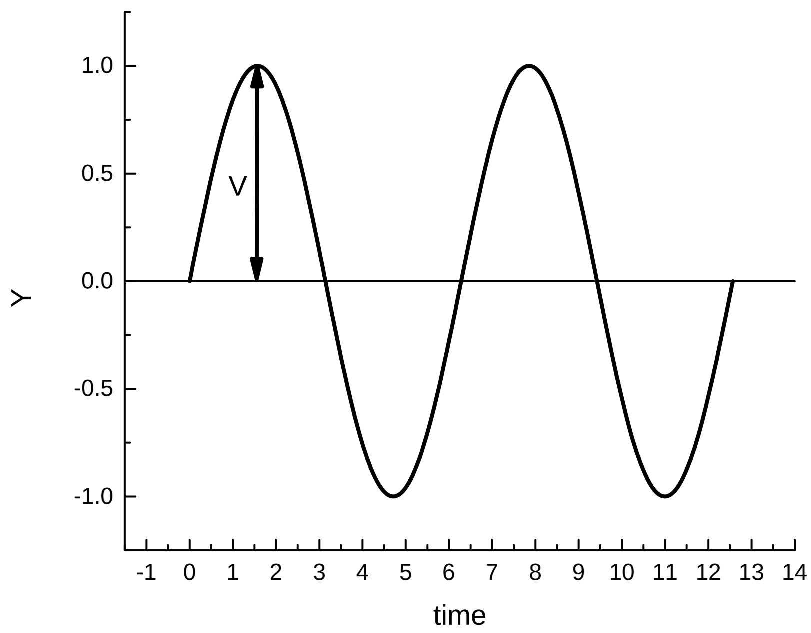
Non-linear RLC circuit

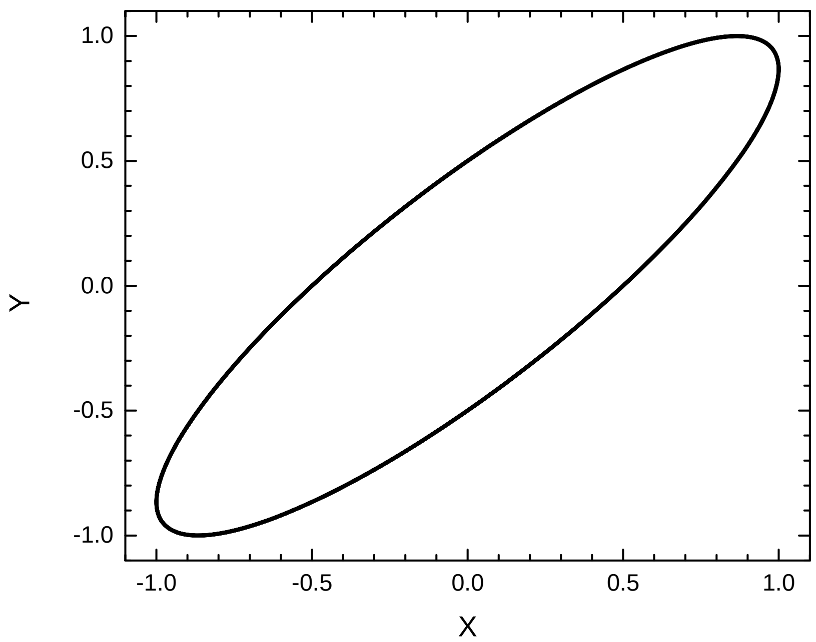




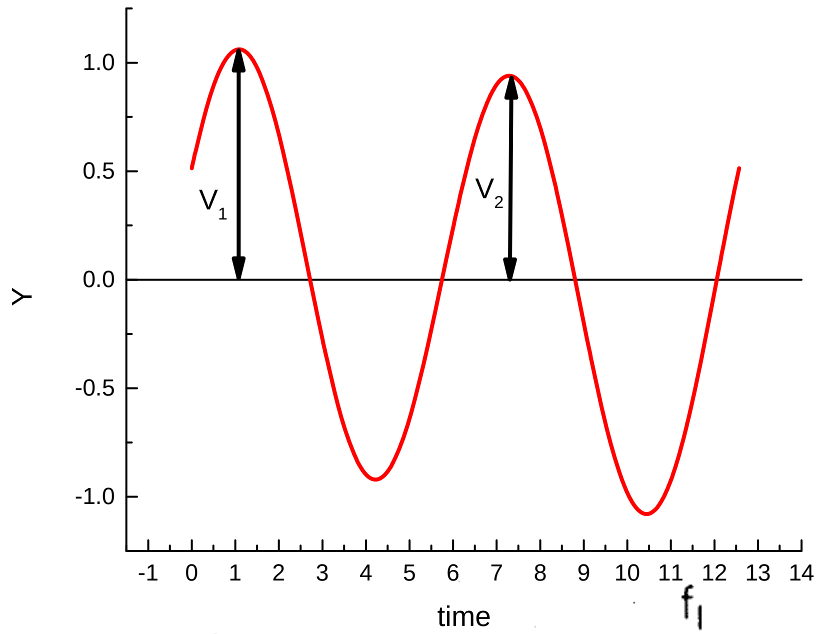
$$C(V)=\frac{C_0}{(1+V/\varphi)^{\gamma}}$$

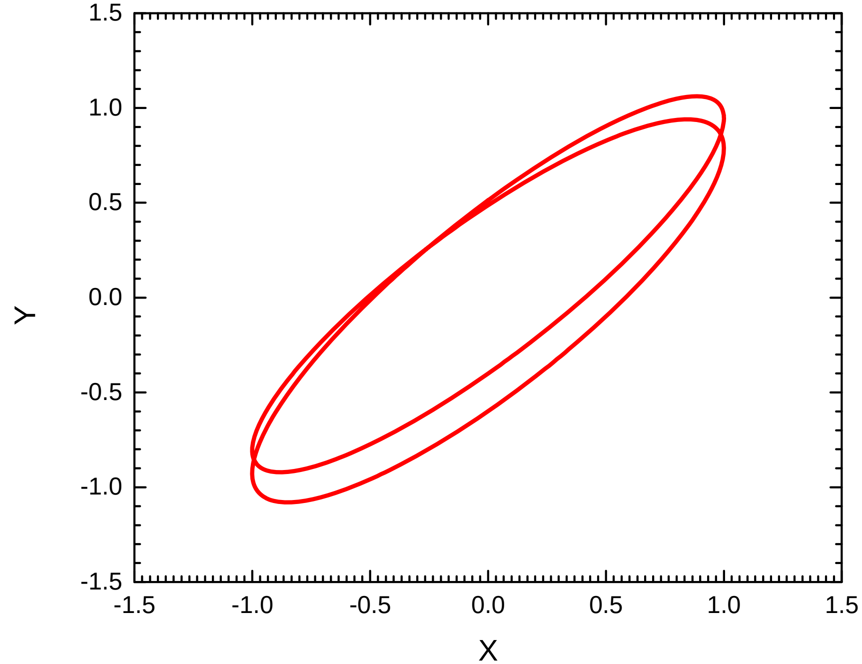
Before bifurcation

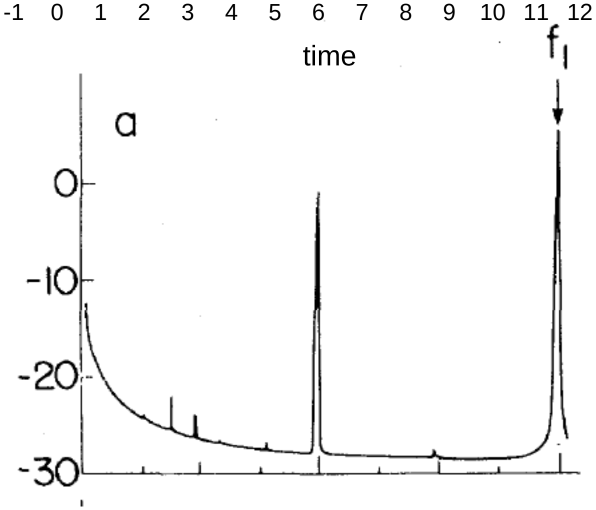




After bifurcation





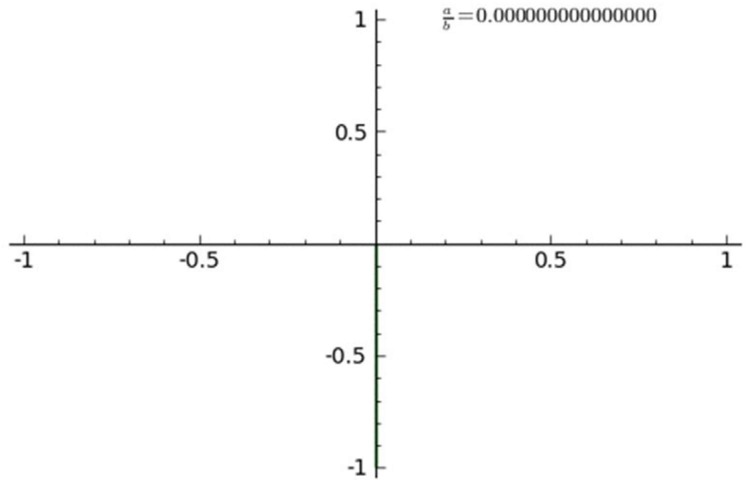


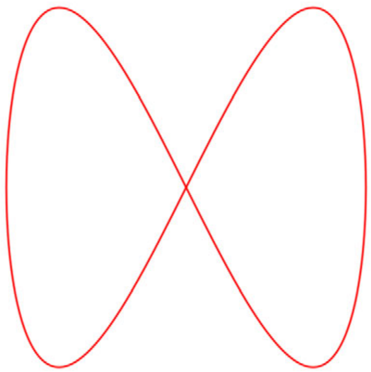
P. Linsey, PhysRevLett.47.1349 (1981).

Lissajous curve

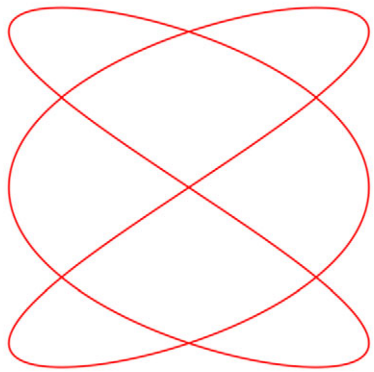
$$x = \sin(at + \delta)$$

$$y = \sin(bt)$$

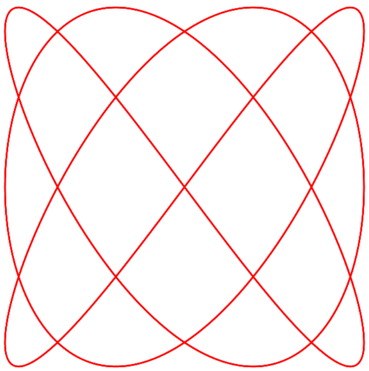




$a/b = 1/2$



$a/b = 3/2$



$a/b = 3/4$