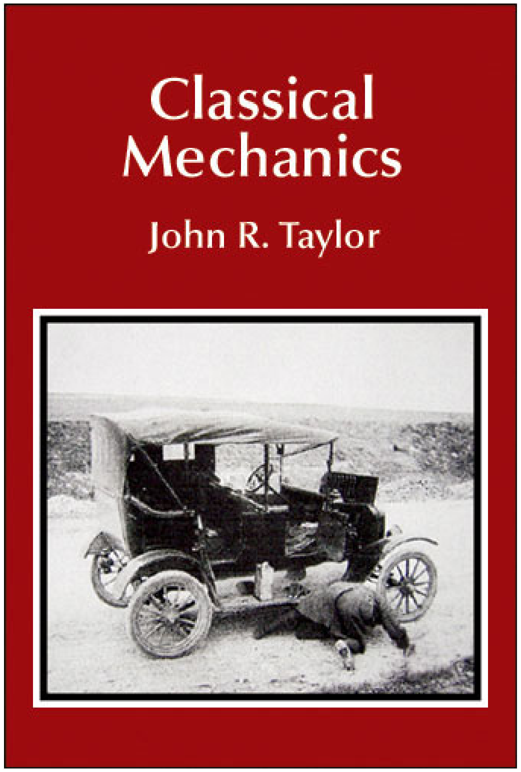
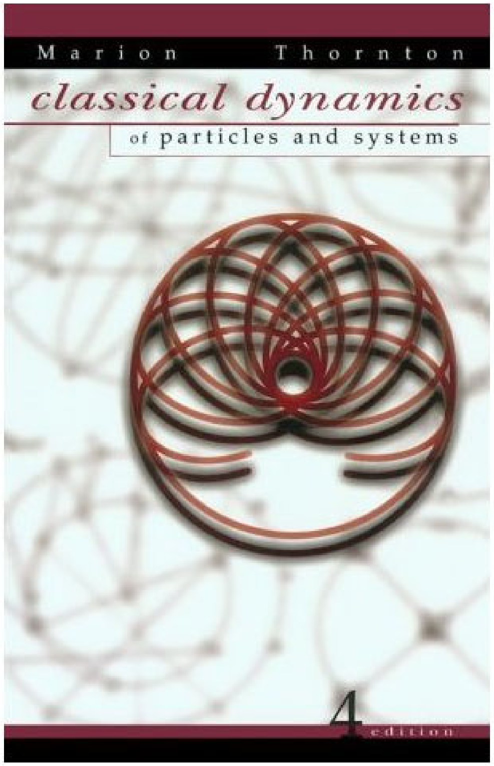
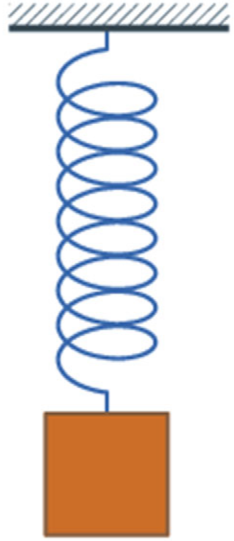
Lab 5: Damped simple harmonic motion

- Simple harmonic oscillation
- Damped harmonic oscillation







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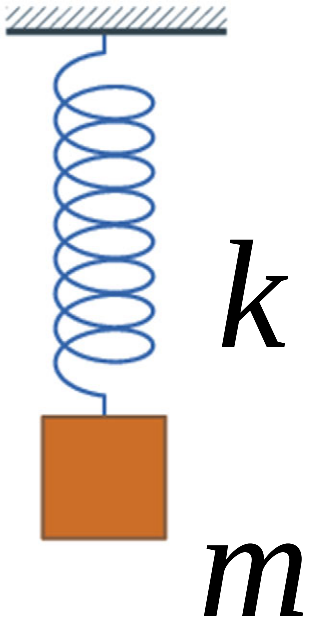
Simple harmonic oscillation

Ideal case: no friction

$$\left.\begin{aligned}\text{Hooke's law: } & F = -kx \\ \text{Newton's 2nd law: } & F = m\ddot{x}\end{aligned}\right\}$$



$$\Rightarrow -kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0 \qquad \omega^2 = \frac{k}{m}$$

$$\ddot{x} + \omega^2 x = 0 \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

solution: $x = A\cos(\omega t + \phi)$

A: Amplitude

ϕ: phase

Simple harmonic oscillation (cont.)

displacement: $x = A\cos\left(\omega t + \phi\right)$

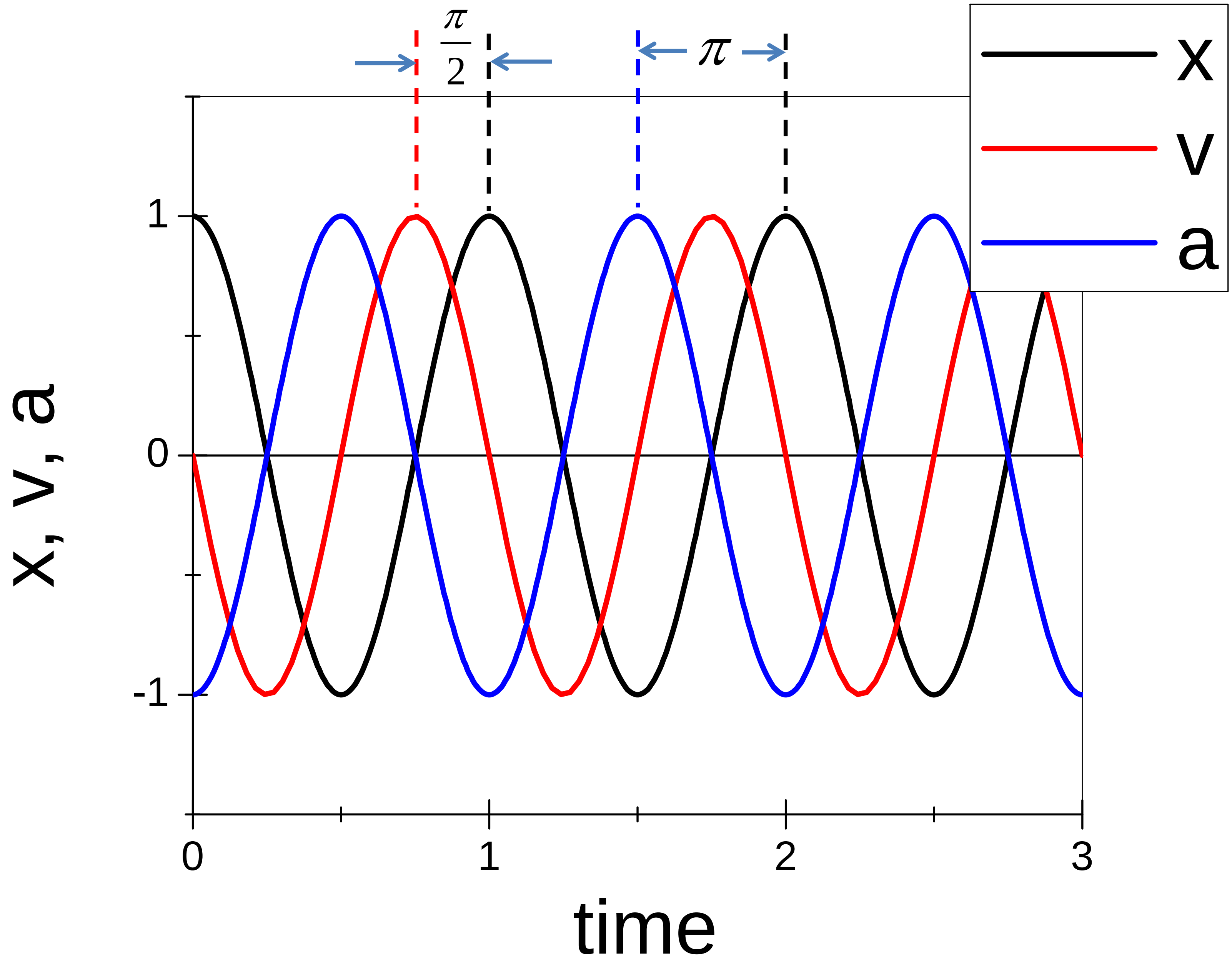
velocity: $v = \dot{x} = -A\omega\sin\left(\omega t + \phi\right)$

$$\Rightarrow v = A\omega\cos\left(\omega t + \phi + \frac{\pi}{2}\right) \qquad \left[\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha\right]$$

accelaration: $a = \ddot{x} = -A\omega^2\cos\left(\omega t + \phi\right)$

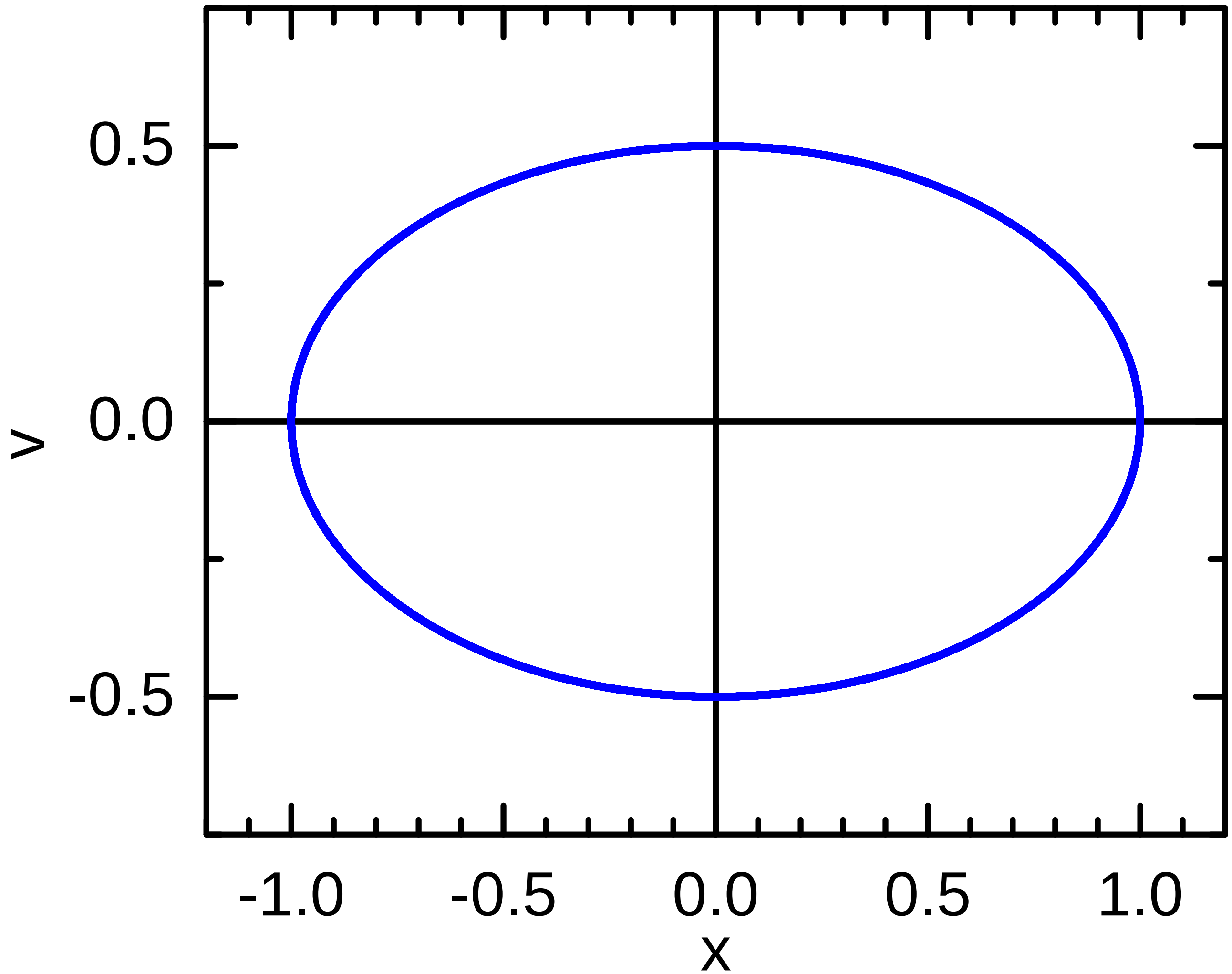
$$\Rightarrow a = A\omega^2\cos\left(\omega t + \phi + \pi\right) \qquad \left[\cos\left(\alpha + \pi\right) = -\cos\alpha\right]$$

Force: $F = ma = Am\omega^2\cos\left(\omega t + \phi + \pi\right) = -kx$

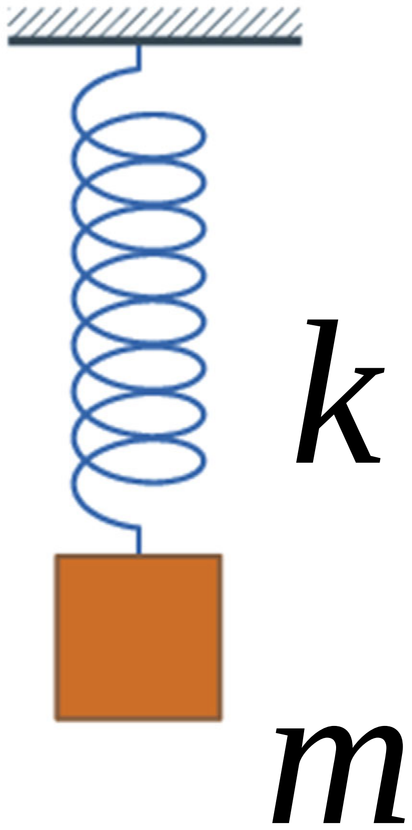


$\frac{\pi}{2}$
π
x
v
a
1
0
-1
x, v, a
0
1
2
3
time

Phase space plot (*v* vs. *x*)



Non-ideal spring (mass)



How should we consider the effect of finite spring mass?

Damped simple harmonic oscillation

Friction: retarding motion (energy dissipation)

$$\left.\begin{array}{r} \text{Hooke's law: } F_1 = -kx \\ \text{Damping force: } \boxed{F_2 = -R\dot{x}} \\ \text{Newton's 2nd law: } F = m\ddot{x} \end{array}\right\}$$

$$m\ddot{x} + R\dot{x} + kx = 0$$

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{k}{m}}, \quad \gamma = \frac{R}{2m}$$

Assume a solution: $x = e^{\lambda t}$

$$\Rightarrow \quad e^{\lambda t}\left(\lambda^2 + 2\gamma\lambda + \omega^2\right) = 0$$

Damped simple harmonic oscillation (cont.)

$$\lambda^2 + 2\gamma\lambda + \omega^2 = 0$$

$$\lambda_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

$$\omega^* = \sqrt{\gamma^2 - \omega^2}$$

$$x = e^{-\gamma t}\left(Ae^{\omega^* t} + Be^{-\omega^* t}\right)$$

underdamping: $\gamma^2 < \omega_0^2$ $\quad \omega^* = i\sqrt{\omega^2 - \gamma^2}$

critical damping: $\gamma^2 = \omega_0^2$ $\quad \omega^* = 0$

overdamping: $\gamma^2 > \omega_0^2$ $\quad \omega^* = \sqrt{\gamma^2 - \omega^2}$

Damped harmonic oscillation (underdamping)

let's define $\omega_1 \equiv \sqrt{\omega^2 - \gamma^2}$

$$\Rightarrow \quad x(t) = e^{-\gamma t}\left(Ae^{i\omega_1 t} + A^* e^{-i\omega_1 t}\right)$$

B is replaced by A^* because x is a real function. let $A = \frac{C}{2}e^{i\phi}$

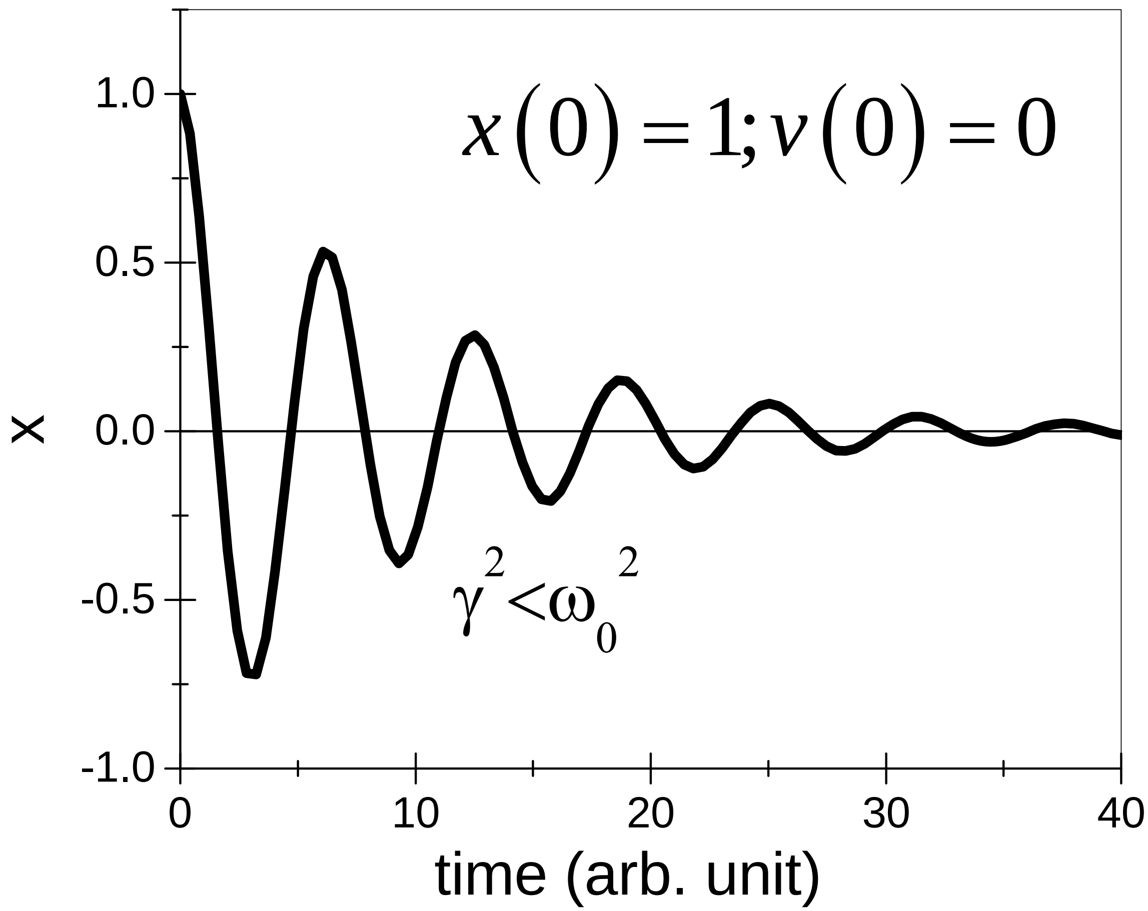
$$\Rightarrow \quad x(t) = \frac{C}{2}e^{-\gamma t}\left(e^{i\omega_1 t + i\phi} + e^{-i\omega_1 t - i\phi}\right)$$

using Eular's formula $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

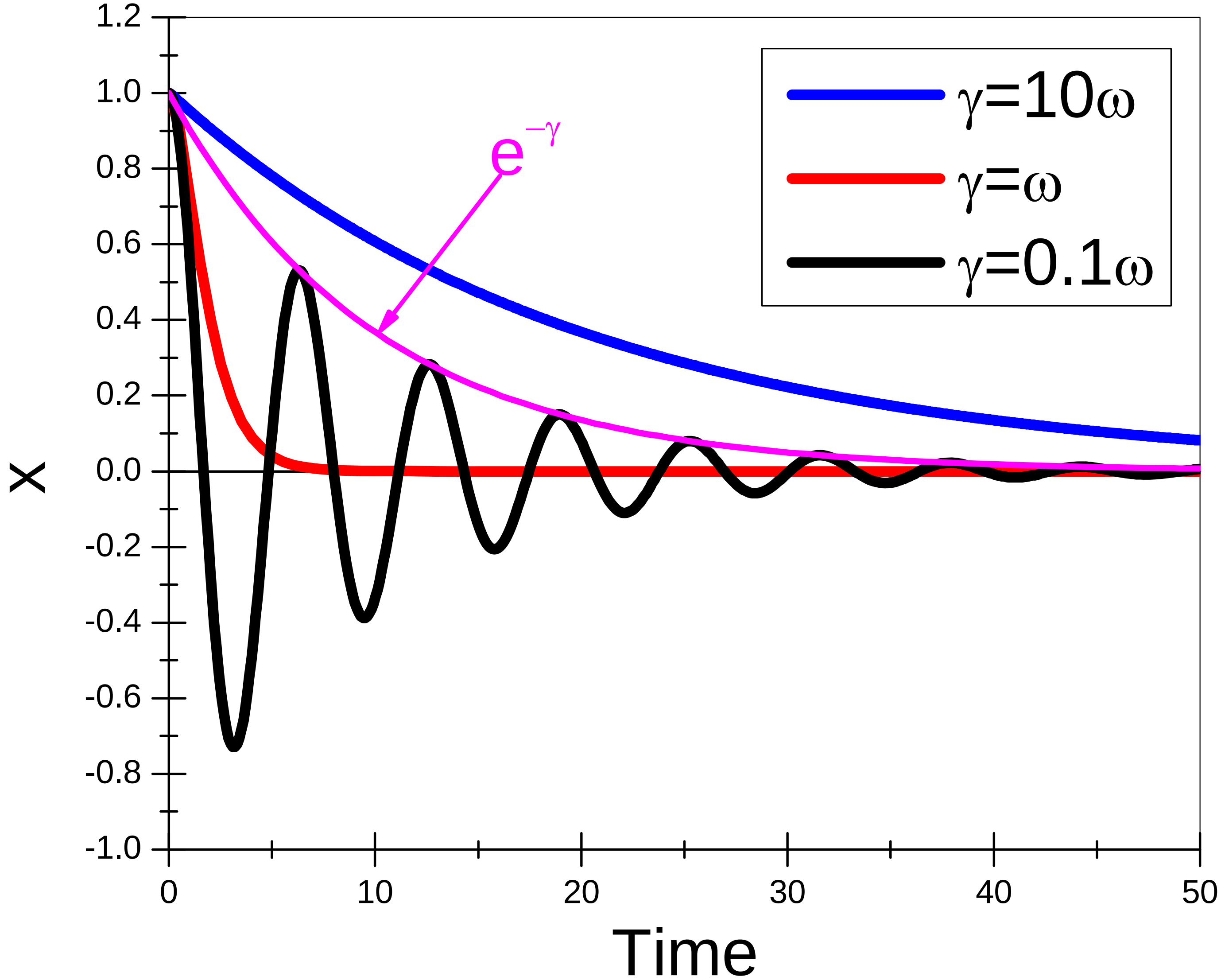
$$\Rightarrow \quad x(t) = Ce^{-\gamma t}\cos\left(\omega_1 t + \phi\right)$$

Damped harmonic oscillation (underdamping)

$$x(t) = Ce^{-\gamma t}\cos(\omega_1 t + \phi) \qquad \omega_1 = \sqrt{\omega^2 - \gamma^2}$$

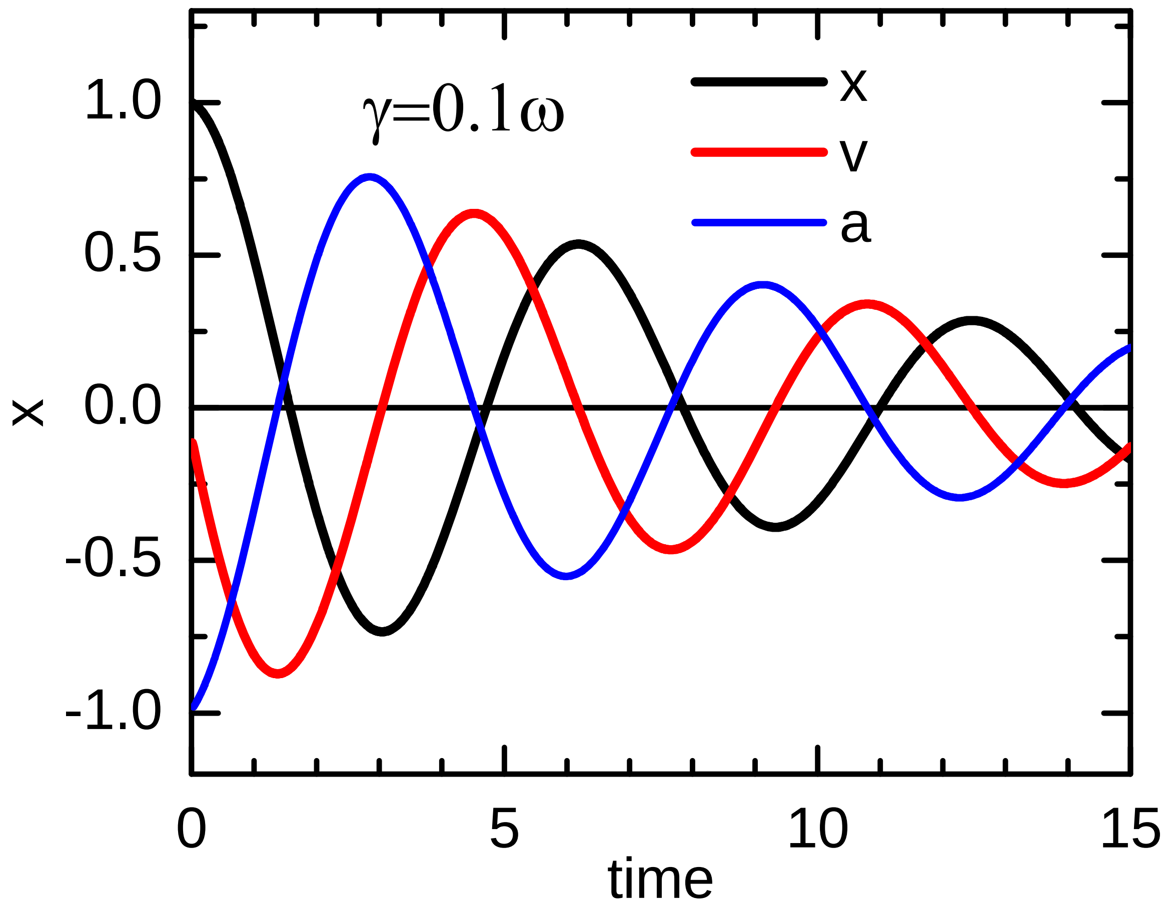


Damped harmonic oscillation



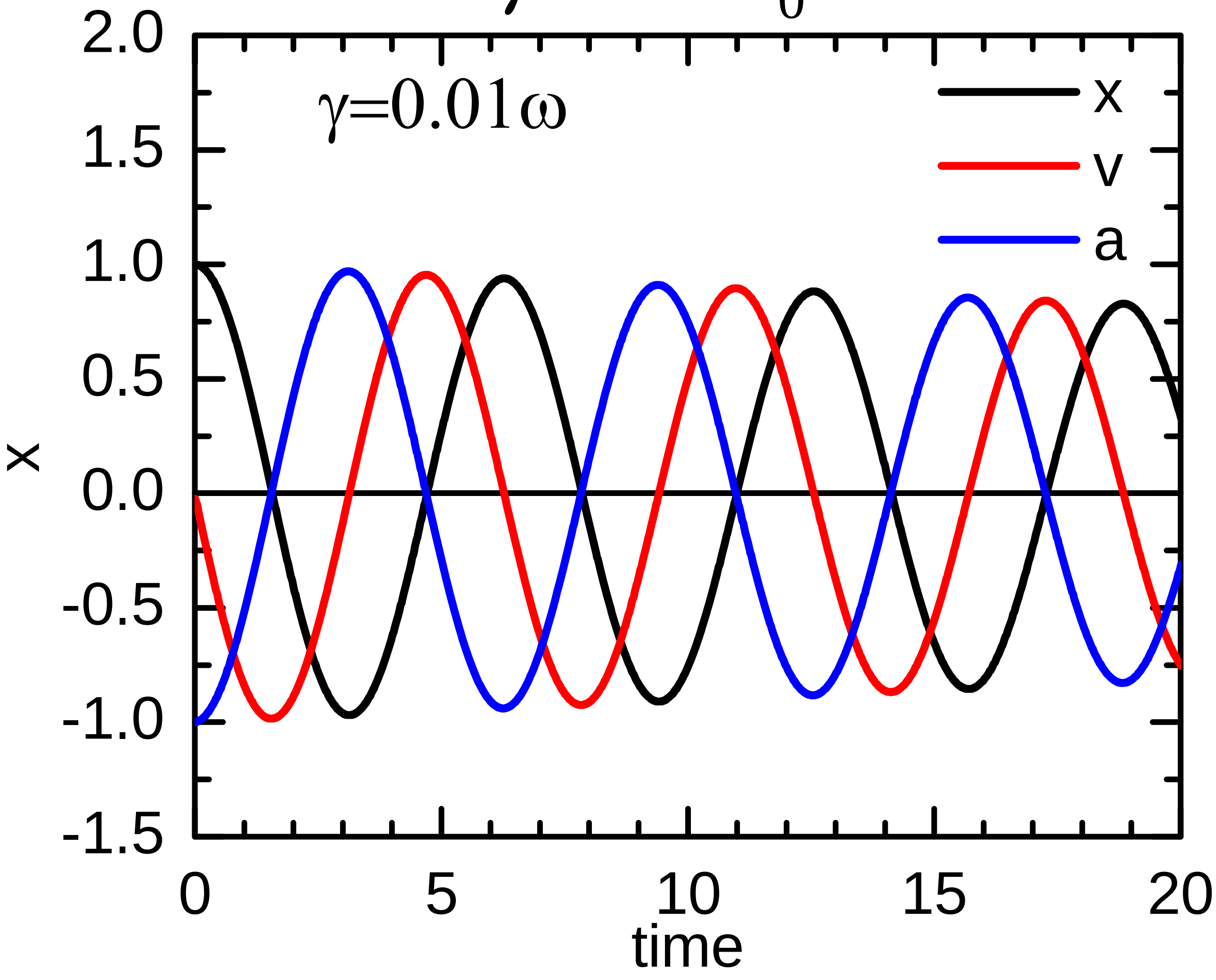
Damped harmonic oscillation (underdamping)

$$x = Ce^{-\gamma t}\cos\left(\omega_1 t + \phi\right)$$

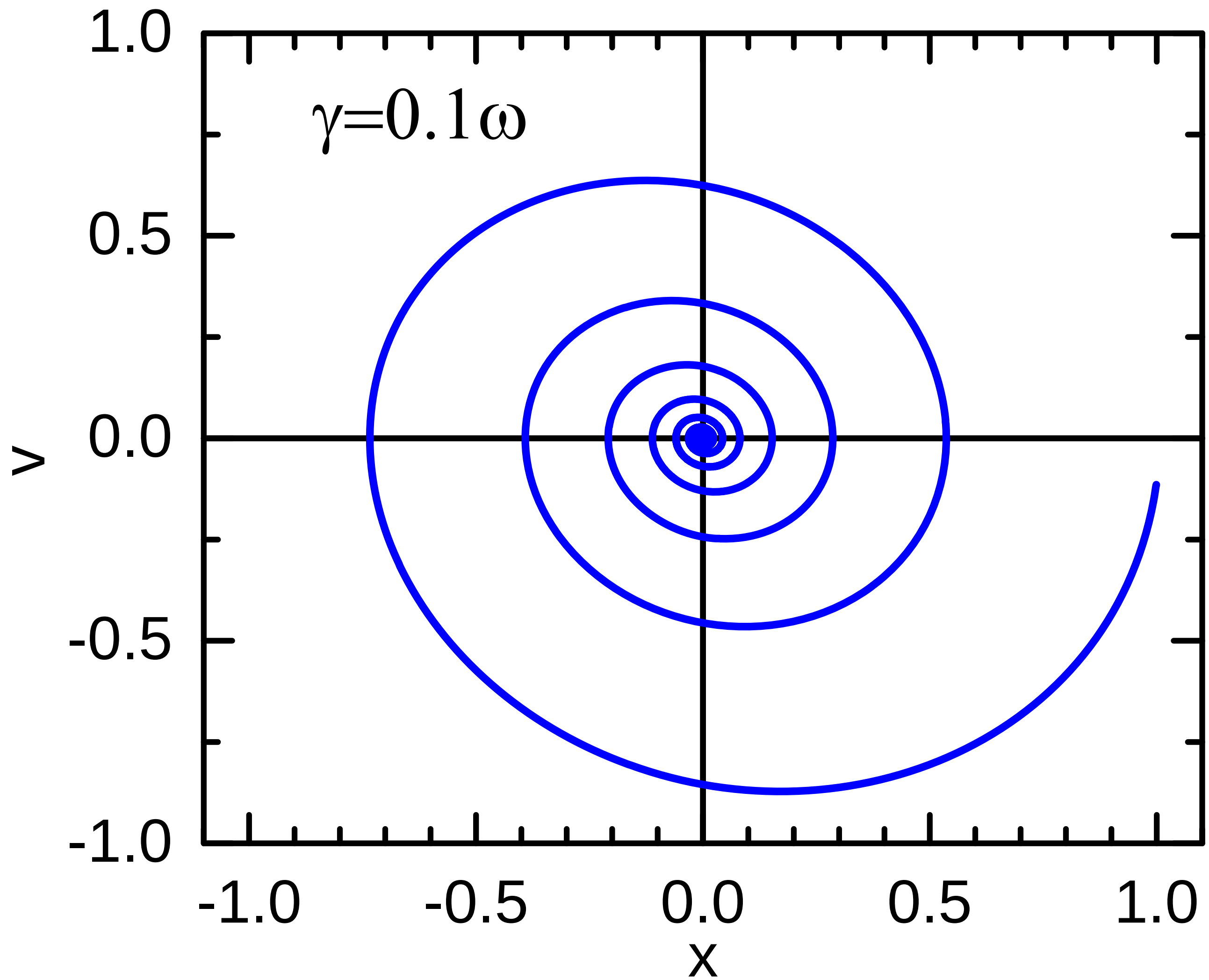


Approaching the ideal limit

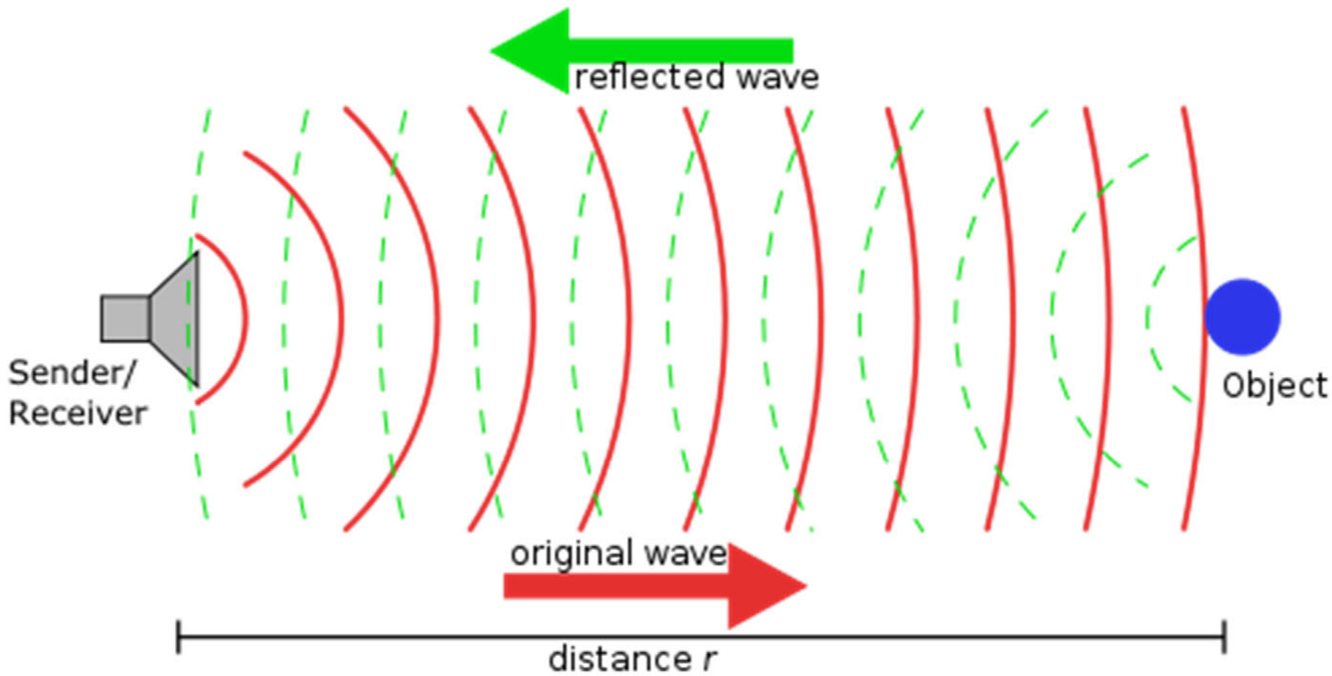
$$\gamma^2 \ll \omega_0^2$$

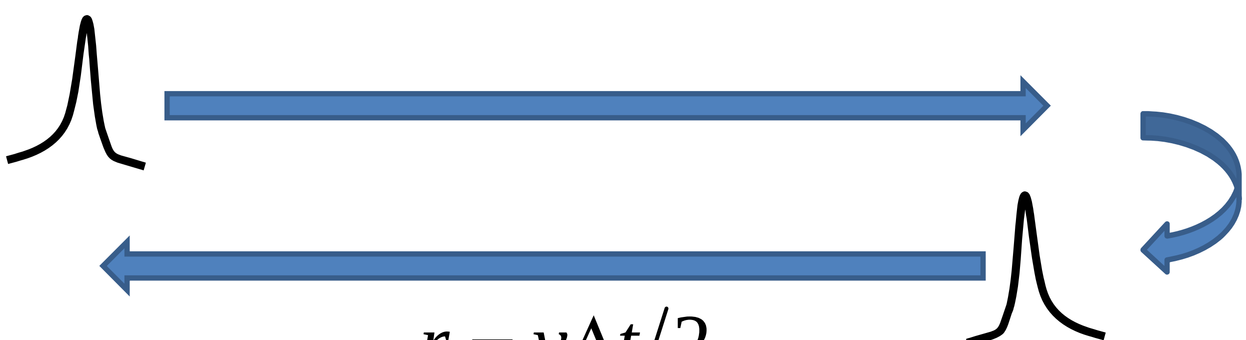


Phase space plot (*v* vs. *x*)



Detection: ultrasonic motion detector





$$r = v\Delta t/2$$