

Damped Simple Harmonic Motion

Purpose

To understand the relationships between force, acceleration, velocity, position, and period of a mass undergoing simple harmonic motion and to determine the effect of damping on these relationships.

Background

When a spring is stretched a distance x from its equilibrium position, according to Hooke's law it exerts a restoring force $F = -kx$ where the constant k is called the spring constant. If the spring is attached to a mass m , then by Newton's second law, $-kx = m\ddot{x}$, where \ddot{x} is the second derivative of x with respect to time. This differential equation has the familiar solution for oscillatory (simple harmonic) motion:

$$x = A \cos(\omega t + \varphi) \quad (1)$$

where A and φ are constants determined by the initial conditions and $\omega = \sqrt{k/m}$ is the angular frequency. The period is $T = 2\pi\sqrt{m/k}$. By differentiating Eq.(1) we determine the velocity $v = -A\omega \sin(\omega t + \varphi)$, which can be rewritten as

$$v = A\omega \cos\left(\omega t + \frac{\pi}{2} + \varphi\right) \quad (2)$$

By differentiating again we obtain the acceleration $a = -A\omega^2 \cos(\omega t + \varphi)$, which can be rewritten as

$$a = A\omega^2 \cos(\omega t + \pi + \varphi) \quad (3)$$

From the acceleration we find the force,

$$F = ma = mA\omega^2 \cos(\omega t + \pi + \varphi) \quad (4)$$

From these four equations we see that the velocity leads the displacement in phase by $\pi/2$ while the force and acceleration lead by π .

For actual oscillating masses the motion is frequently not quite this simple since frictional forces act to retard the motion. To treat friction we will assume that the retarding force is proportional to the speed of the mass so that Newton's second law becomes

$$m\ddot{x} = -kx - R\dot{x} \quad (5)$$

where R is the frictional coefficient. We assume a solution $x = e^{\lambda t}$ which we substitute into Eq.(5) to obtain

$$m\lambda^2 e^{\lambda t} = -ke^{\lambda t} - R\lambda e^{\lambda t} \Rightarrow me^{\lambda t} \left(\lambda^2 + \frac{R}{m}\lambda + \frac{k}{m} \right) = 0 \Rightarrow \lambda^2 + 2\gamma\lambda + \omega^2 = 0$$

where $\gamma = \frac{R}{2m}$. This equation has two roots: $\lambda_{\pm} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$.

The general solution to Eq.(5) can then be written as

$$x = e^{-\gamma t} (Ae^{\omega^* t} + Be^{-\omega^* t}) \quad (6)$$

where $\omega^* = \sqrt{\gamma^2 - \omega^2}$, while A and B are constants determined by initial conditions. Here we are only interested in the case where the frictional forces are fairly small so that $\gamma < \omega$, which is described as “underdamping”. In this case, ω^* is an imaginary quantity and can be rewritten as $\omega^* = i\sqrt{\omega^2 - \gamma^2} = i\omega_1$, where ω_1 is a real quantity. Using Euler's identity, $[2 \cos x = e^{ix} + e^{-ix}]$, Eq.(6) can be written as

$$x = Ce^{-\gamma t} \cos(\omega_1 t + \varphi) \quad (7)$$

where C and φ are constants determined by the initial conditions (i.e. A and B). The interpretation of this result is very easy. The motion is oscillatory with an angular frequency $\omega_1 = \sqrt{\omega^2 - \gamma^2}$, slightly smaller than it would be if there were no damping. This shift will be too small to detect for the small value of γ that we will be using in the experiment. In addition the amplitude of the oscillation decreases exponentially with a damping coefficient $\gamma = \frac{R}{2m}$.

Equipment

Computer
 LabPro interface
 Motion detector
 Force transducer
 Masses (.1 to 0.5 kg)
 Mass holder
 Spring
 Paper plate

Procedure

In this experiment you will hang a weight from the bottom of a spring and attach the spring to a force transducer. Below the weight you will place an ultrasonic position detector. As the spring oscillates the program Logger Pro records the time dependence of the force the spring exerts on the force transducer and of the position of the weight above the position detector. By using the computer to rapidly and precisely record the data, you will be able to make careful comparisons between the data and theory.

1. Connect the force transducer to the analog CH1 plug on the LabPro interface. Connect the ultrasonic position sensor to the DIG/SONIC1 connector on the LabPro interface. Turn on the computer; the LabPro interface should always be on.
2. Start Logger Pro and follow the instructions in the Hints appendix below. The data sampling rate should be 30 samples per second.
3. Determination of k : to determine k we will use the Hooke's law: $\Delta F = -k\Delta x$, here the force ΔF is obtained by using calibrated weights with stamped mass values and gravity constant $g=9.80 \text{ m/s}^2$ (i.e. $\Delta F = mg$). Hang the spring and weight holder from the force transducer. Use Logger Pro to measure the stretches of the spring as you increase the weight on the holder in 100 g increments. For each measurement make sure that the weight is completely at rest. Check that change in displacement given by Logger Pro agrees with the meter stick measurement. Also compare the force you obtained from the mass with the force measured by the force transducer. Use ORIGIN or Excel to plot force vs. displacement and find the slope (which is the spring constant k !) of the line. Be very careful not to let the weights slip off the weight holder since the position detector is right below.
4. Put about 500 g on the weight holder without the paper plate and use Logger Pro to measure the period of oscillation of the system. Record the position and force vs. time for several oscillations of the weight. Allow Logger Pro to calculate columns for velocity and acceleration. To obtain accurate data, choose the fastest data rate that will give you reliable data. [If the rate is too high or the sensor is too close to the weight, the ultrasonic transducer will not have time to stop oscillating before the returning echo arrives; there will be missing entries in the data list.]
5. Import the data into ORIGIN. Plot the position x vs. time t , velocity v vs time t , and acceleration a vs time t , making sure that the values oscillate about a mean of zero.
6. You will find the period T most accurately by locating the times where the curves pass through zero. To calculate the predicted (theoretical) period from the equation $T = 2\pi\sqrt{m/k}$, use your experimental value for k and the mass of the oscillating weight. Be sure to include the mass of the weight holder. Q1: Should you include the mass of the spring? Think carefully about it. Does the spring move with the weight? In particular, does the bottom part of the spring move? Does the top part move? So what do you suggest we do with the mass of the spring? Note that your theoretical value for T has an error due to the uncertainty in both k and m . Compare your prediction with the experimental value of T . Q2: Do the two agree within the error limits you have determined? If not discuss possible sources of the discrepancy.
7. From the graphs of x , v , and a vs. t **accurately** determine the relative phases of oscillation (phases of v and a with respect to the phase of x) and compare with the theory. To do this determine the times when each of these quantities pass through

zero and convert the difference in times into a phase shift [$\Delta\phi = 2\pi\Delta t / T$]. Report your methods. What relative phase do you expect between the force and position oscillations? Is this seen on a graph of force vs t ?

8. Plot the velocity as a function of position. This is known as a phase space plot. Explain the shape of the orbit you see. Q3: Derive a theoretical expression for the shape of this orbit. [Hint: use the first two equations in the BACKGROUND section.]
Q4: What is the orbit in phase space of the oscillator when the mass is at rest?
9. Without extra damping, the oscillating weight is very slowly damped by air resistance to the motion. In order to study damping effects, increase the damping by placing a light foam plate, or a piece of paperboard on the weight holder. First use the position detector to determine the equilibrium position of the weight. Then set the weight in motion and record the amplitude of oscillation over enough cycles until the peak amplitude has decayed by at least 50% during the measurement. Release the weight carefully so that it doesn't sway from side to side. The theory predicts that for underdamped oscillations, the peak amplitude (A) will decrease exponentially, provided that the friction force is proportional to the velocity (viscous damping). Extract the peak amplitude (of each oscillation) from your data. (briefly explain how you did it) **Subtract off the equilibrium position** and plot the **natural log** of the peak amplitude vs. time (A vs. t). Is it a straight line? If not, why not? Do you expect the air resistance to be of the viscous kind, or turbulent? Assuming it is an exponential decay, make a linear least square fit to obtain the damping constant γ .
Q5: Is the relation $\gamma \ll \omega$ satisfied? Again plot the velocity as a function of position for the oscillation with extra damping. Explain your observations.
10. The results section of your report will include your data plots, your results for k , T , γ , and the relative phases, including uncertainties for all quantities. Provide answers to all labeled questions (Q1, Q2, etc., show the questions numbers in your report) and any other questions and instructions that are not labeled.

HINTS FOR USING LOGGER PRO

To setup the program, go to:

Menu EXPERIMENT

o CONNECT INTERFACE

o CONNECT ON PORT: select COM1

Button LABPRO

o Drag MOTION DETECTOR to DIG/SONIC1 box

o Drag DUAL RANGE FORCE to CH1 box (analog channels)

o CLOSE

Menu EXPERIMENT

- o DATA COLLECTION ...
 - o Tab COLLECTION
 - o LENGTH 10 seconds
 - o Sample at Time Zero: off
 - o SAMPLING RATE 30 samples per sec
 - o DONE
- To collect data, click button COLLECT.
- To export data, go to:
- Menu FILE
- o EXPORT DATA AS TEXT.