Intermediate Quantum Mechanics
Lecture 9 Notes (2/18/15)
The Ehrenfest Theorem

Energy eigenstates

- The operator $\hat{H}$ that generates time translations: $\hat{U}(\epsilon) = \hat{I} - i\epsilon\hat{H}/\hbar$ is Hermitian. That means that it corresponds to an observable. That observable is the energy and $\hat{H}$ is the energy operator. For now, there is no reason to accept this just as there is no reason to accept that $\hbar\hat{k}$ is the momentum operator. We will see the reasons soon.

- Since $\hat{H}$ is a Hermitian operator, its eigenstates form a complete basis. Any state can be expand in the terms of the energy eigentstates.

$$|\psi(t)\rangle \;=\; \sum_i a_i(t)\,|E_i\rangle$$

  Here I've taken the energy eigenstates to be discreet. If they were continuous we would have instead:

$$|\psi(t)\rangle \;=\; \int a(E',t)\,|E'\rangle\,dE'$$

- The time derivative of $|\psi(t)\rangle$ is:

$$\frac{d}{dt}\,|\psi(t)\rangle \;=\; \sum_i \dot{a}_i(t)\,|E_i\rangle$$

  where I'm using the notation used in Classical Mechanics that: $\dot{a}(t) = \dfrac{da(t)}{dt}$

- Combining this with the Schrodinger equation:

$$\frac{d}{dt}\,|\psi(t)\rangle \;=\; -\frac{i}{\hbar}\,\hat{H}\,|\psi(t)\rangle \;=\; -\frac{i}{\hbar}\,\hat{H}\sum_i a_i(t)\,|E_i\rangle$$
$$=\; -\frac{i}{\hbar}\sum_i a_i(t)\hat{H}\,|E_i\rangle \;=\; -\frac{i}{\hbar}\sum_i a_i(t)E_i\,|E_i\rangle$$

  we have:

$$\sum_i \dot{a}_i(t)\,|E_i\rangle \;=\; -\frac{i}{\hbar}\sum_i a_i(t)E_i\,|E_i\rangle \qquad \Rightarrow \qquad \dot{a}(t) \;=\; -\frac{i}{\hbar}\,a_i(t)E_i$$

$$\Rightarrow \qquad a_i(t) \;=\; a_i(0)e^{-iE_it/\hbar}$$

  Then: $$|\psi(t)\rangle \;=\; \sum_i a_i(0)e^{-iEt/\hbar}\,|E_i\rangle$$

- If you want to know the time dependence of a state, it is often useful to expand the state in terms of the energy eigenstates since the time dependence of the coefficients of the energy eigenstates is very simple: $a_i(t) \;=\; a_i(0)e^{-iE_it/\hbar}$.

The Ehrenfest Theorem

- We will now derive a very important theorem that will allow us to make the connection between quantum mechanics and classical physics.
- Let $\hat{A}$ be any Hermitian operator. The average value of its observable as a function of time for system that is in the state $|\psi(0)\rangle$ at time $t = 0$ is given by the expectation value: $\overline{A}(t) \;=\; \langle\psi(t)|\hat{A}\,|\psi(t)\rangle$.
- We can then use the chain rule to find the rate of change of the average value of A. Note that the operator$\hat{A}$ itself does not depend on t. All of the time dependence is in the state $|\psi(t)\rangle$.

$$\dot{\overline{A}}(t) \;=\; \left(\frac{d}{dt}\,\langle\psi(t)|\right)\hat{A}\,|\psi(t)\rangle \;+\; \langle\psi(t)|\hat{A}\left(\frac{d}{dt}\,|\psi(t)\rangle\right)$$

Using: $$\frac{d}{dt}\,\langle\psi(t)| \;=\; \frac{i}{\hbar}\,\langle\psi(0)|\hat{H}^\dagger \;=\; \frac{i}{\hbar}\,\langle\psi(0)|\hat{H}$$

$$\dot{\overline{A}}(t) \;=\; \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A}\,|\psi(t)\rangle \;-\; \frac{i}{\hbar}\langle\psi(t)|\hat{A}\hat{H}\,|\psi(t)\rangle \;=\; \frac{i}{\hbar}\langle\psi(t)|\,[\hat{H},\hat{A}]\,|\psi(t)\rangle$$

- The Ehrenfest Theorem relates the time derivative of the expectoration value of an operator to the expectation value of the commutator of the operator with the Hamiltonian.

$$\frac{d}{dt}\,\langle\psi(t)|\hat{A}\,|\psi(t)\rangle \;=\; -\frac{i}{\hbar}\,\langle\psi(t)|\,[\hat{A},\hat{H}]\,|\psi(t)\rangle$$

- A corollary is that the average value of an observable is conserved, $\dot{\overline{A}}(t) \;=\; 0$ if the operator commutes with the Hamiltonian $[\hat{A},\hat{H}] \;=\; 0$