Intermediate Quantum Mechanics
Lecture 4 Notes (2/2/15)

Example of Electron Spin

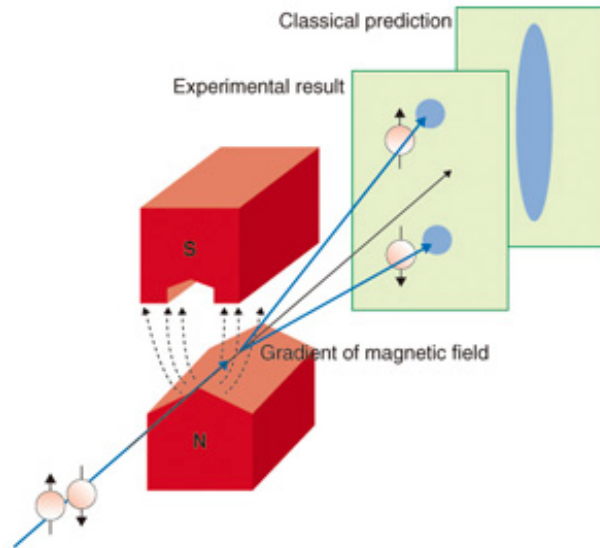**Electron Spin**

- The postulates given in the Lecture 3 Notes represent all the physics of quantum mechanics.

- We will now give a concrete example of the use of these postulates for the simplest nontrivial system possible, a system who's states are elements of a 2-dimensional vector space. We will use the example of the spin of an electron.

- The electron is an elementary (point-like) particle. It has three intrinsic properties: mass, electric charge and angular momentum (spin). It is hard to grasp how a point particle with no size and therefore no moment of inertia can have an angular momentum. Nevertheless, the electron carries intrinsic angular momentum completely separate from any orbital angular momentum that it might have due to its motion. You could think of the electron in analogy to a spinning top but this isn't really helpful. To me, the electron having intrinsic angular is no stranger than its having intrinsic mass or electric charge. The only difference is that mass and charge are scalars while angular momentum is an arrow (it has a direction).

- The physics of quantum mechanics is completely given by the postulates of the previous lecture. The spin of the electron has to do with the physics of elementary particles. It happens that we live in a universe where some of the elementary particle have spin. You can imagine a universe in which quantum mechanics applies but in which the electron doesn't have spin. It just wouldn't be a very hospitable universe.

- In classical physics, a spinning charged top is a magnetic dipole with a magnetic moment. It turns out that the spinning electron also has a magnetic moment, $\vec{\mu}$. Since the charge of the electron is negative, the spin and the direction of the magnetic dipole moment arrow are opposite. For me, this is very confusing but we have to deal with it. Thank you Mr. Franklin.

- The energy of a magnetic dipole situated in a region of magnetic field is: $U = -\vec{\mu} \cdot \vec{B}$. The energy is minimum when the dipole moment and the magnetic field are aligned. That means that the magnetic energy of the electron will be minimum when its spin is anti-aligned (opposite in direction) to the magnetic field.

Stern-Gerlach Device

- A Stern-Gerlach device is a magnet with one pole piece more pointed than the other. The magnitude of the magnetic field will be greater near the pointed pole piece resulting in a gradient of magnetic field between the two pole pieces.

- An electron with component of its spin in the direction of the magnetic field will experience a force deflecting it toward the region of weaker magnetic field while an electron with component of its spin opposite the direction of the magnetic field will experience a force deflecting it toward the region of stronger magnetic field. This is shown in the figure below.

- If a beam of electrons is sent into the device, classically, we would expect a continues spray of electrons to emerge since the component of spin of the electron in the direction of the field can be anything between its magnitude and the negative of its magnitude.

- If we do the experiment, we find that only two distinct beams of electrons comes out, one corresponding to electrons with spin "up" and the other to electrons with spin "down".

- For our discussion, I will use a schematic Stern-Gerlach device as shown here.
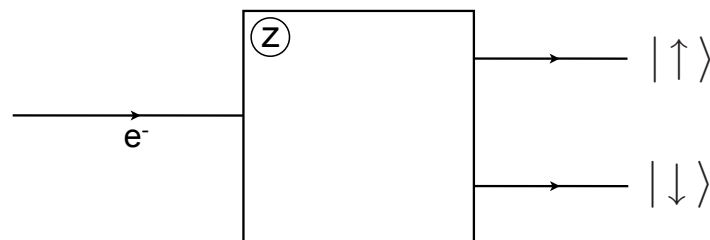
  The z in the upper corner indicates that the magnetic field gradient is oriented in the z-direction. This splits the electrons into two beams in z with one beam corresponding to electrons with spin-up in the z-direction and the other to electrons with spin-down in the z-direction.

Applying the postulates of QM
- When we send an electron into a z-oriented Stern-Gerlach device, we are making a measurement of (observing) the spin in the z-direction. The measured observable is $+1$ if the electron is spin-up in the z-direction and -1 if the if

the electron is spin-down in the z-direction. By Postulate 2, there must be a Hermitian corresponding to observing the spin in the z-direction. We'll call this operator $\hat{\sigma}_z$.

$$\hat{\sigma}_z \,|\uparrow\,\rangle \;=\; |\uparrow\,\rangle \qquad\qquad \hat{\sigma}_z \,|\downarrow\,\rangle \;=\; -|\downarrow\,\rangle$$

- Since $\hat{\sigma}_z$ has two eigenvectors, $|\uparrow\,\rangle$ and $|\downarrow\,\rangle$, the spin state of the electron is a vector in a 2-dimensional vector space. By Postulate 1, any state $|\psi\rangle$ of the electron spin can then be written as a linear superposition of the eigenvector of $\hat{\sigma}_z$:
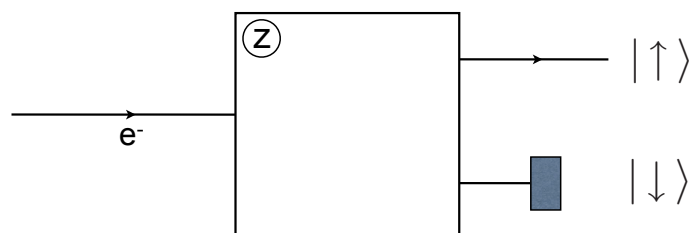
$$|\psi\rangle \;=\; \alpha\,|\uparrow\,\rangle + \beta\,|\downarrow\,\rangle$$

- If we send an electron in the state $|\psi\rangle = \alpha\,|\uparrow\,\rangle + \beta\,|\downarrow\,\rangle$ into a z-oriented Stern-Gerlach device, the electron will emerge in either the $|\uparrow\,\rangle$ state or the $|\downarrow\,\rangle$ state. By Postulate 4, the probability of the electron to be observed in the $|\uparrow\,\rangle$ state is $|\alpha|^2$ and the probability to be observed in the $|\downarrow\,\rangle$ state is $|\beta|^2$. If we send a beam of electrons all in the $|\psi\rangle = \alpha\,|\uparrow\,\rangle + \beta\,|\downarrow\,\rangle$ state into the device, a fraction $|\alpha|^2$ will emerge in the $|\uparrow\,\rangle$ state and a fraction $|\beta|^2$ will emerge in the $|\downarrow\,\rangle$ state.
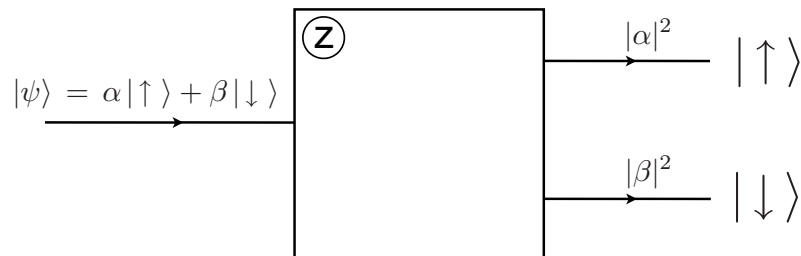
Preparing a state

- We can use a modified Stern-Gerlach device to prepare an electron in a particular state. For example, if we use a z-oriented Stern-Gerlach device but block the outgoing $|\downarrow\,\rangle$ beam the electrons we keep will all be in the $|\uparrow\,\rangle$ state.


Repeatable of measurement

- If we take an electron in the $|\uparrow\,\rangle$ state and send it through a z-oriented Stern-Gerlach device, it is certain to emerge in the $|\uparrow\,\rangle$ state. We can repeat this over and over again. As long as we don't "disturb" the electron between measurements, it will remain in the $|\uparrow\,\rangle$ state.

- It is important that this happen, since otherwise saying that an electron is in a particular state, for example $|\uparrow\,\rangle$, wouldn't have meaning.

Spin in the x-direction

- If we send an electron in any state into a Stern-Gerlach device oriented in the x-direction, the electron will again emerge in one of two states: spin-up in the x-direction, $|\rightarrow\rangle$, or spin-down in the x-direction, $|\leftarrow\rangle$.
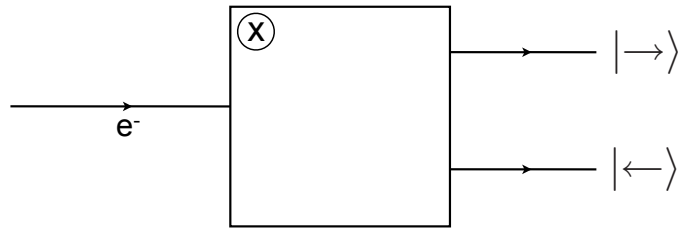
- We can prepare an electron in the $|\rightarrow\rangle$ state by blocking the $|\leftarrow\rangle$ beam in an x-oriented Stern-Gerlach device.
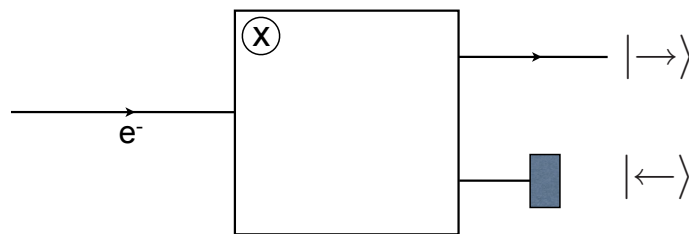
- If we then send an electron in the $|\rightarrow\rangle$ state into a z-oriented Stern-Gerlach device, it will emerge in either the $|\uparrow\rangle$ state or the $|\downarrow\rangle$ state. By symmetry, the probability of each must be a half, $|\alpha|^2 = |\beta|^2 = 1/2$.
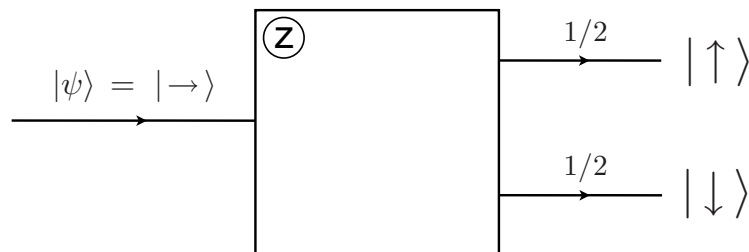
- We can arbitrarily choose the complex phases of α and β such that $\alpha = \beta = 1/\sqrt{2}$.

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

- The state $|\leftarrow\rangle$ must be orthogonal to $|\rightarrow\rangle$ since these are two distinct states distinguished by the the fact that they are states with different eigenvalues of $\hat{\sigma}_x$. $\langle\leftarrow|\rightarrow\rangle = 0$. We then have:

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

Basis representation

- So far we have been working with the states:

4

$$|\uparrow\rangle \qquad |\downarrow\rangle \qquad |\rightarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle \qquad |\leftarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

and the operators: $\hat{\sigma}_z$ and $\hat{\sigma}_x$:

$$\hat{\sigma}_z|\uparrow\rangle = |\uparrow\rangle \qquad\qquad\qquad \hat{\sigma}_z|\downarrow\rangle = -|\downarrow\rangle$$

$$\hat{\sigma}_x|\rightarrow\rangle = |\rightarrow\rangle \qquad\qquad\qquad \hat{\sigma}_x|\leftarrow\rangle = -|\leftarrow\rangle$$

independent of basis representation.

- Now we choose the representation in which $|\uparrow\rangle$ and $|\downarrow\rangle$ are the basis vectors. We could of course choose a different basis, for example, $|\rightarrow\rangle$ and $|\leftarrow\rangle$ but it is conventional to choose the $|\uparrow\rangle$, $|\downarrow\rangle$ basis.

- In this basis, the states and operators are represented as:

$$|\uparrow\rangle \implies \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad\qquad\qquad |\downarrow\rangle \implies \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\rightarrow\rangle \implies \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad\qquad |\leftarrow\rangle \implies \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{\sigma}_z \implies \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad\qquad\qquad \hat{\sigma}_x \implies \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Degrees of Freedom

- All of the parameters in the representations of the vector states and the operators above are real. As long as we are concerned only with states with spin in a direction in the x-z plane, real vectors are sufficient.

- The direction of an arrow in a plane is specified by one parameter usually chosen to be the angle θ that the arrow makes with the z-axis. On the other hand, a general 2-dimensional real vector state has one degree of freedom. We can express that state as

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \qquad\qquad \text{with } \alpha \text{ and } \beta \text{ real.}$$

There are two real numbers but one degree of freedom is removed due to the constraint that $\alpha^2 = \beta^2$. Once α is chosen, β is fixed.

- One parameter is needed to specify a particular direction and there is one degree of freedom. So, the the state with spin-up in any direction in the x-y plane can be specified by the real vector $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$.

- In order to specify a direction in 3-dimensional physical space, two parameters are needed. We need to specify the angle with respect to the z-axis, the zenith angle θ and the angle around the z-axis, the azimuthal angle ϕ
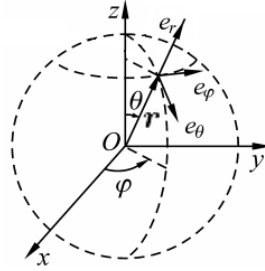
This cannot be done using 2-dimensional real vector states.

Spin in the y direction

- Now take a state prepared to be spin-up in the z-direction, $|\uparrow\rangle$, and pass it through a Stern Gerlach device aligned with its gradient in the y-direction. There are of course tow possible states for the electron that emerges just as there were for a z or x Stern-Gerlach device.
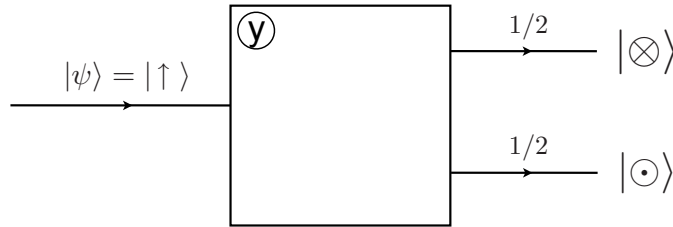
- The states $|\otimes\rangle$ and $|\odot\rangle$ of spin-up and spin-down along y, respectively, must be orthogonal and they must not be multiples of any of the states: $|\uparrow\rangle$, $|\downarrow\rangle$, $|\rightarrow\rangle$ and $|\leftarrow\rangle$.

- We cannot write these as real vectors we must use complex vectors.

- The operator corresponding to measuring the spin in the y direction is $\hat{\sigma}_y$ for which $|\otimes\rangle$ and $|\odot\rangle$ are eigenvectors with eigenvalues 1 and -1, respectively.

$$\hat{\sigma}_y\,|\otimes\rangle \;=\; |\otimes\rangle \qquad\qquad \hat{\sigma}_y\,|\odot\rangle \;=\; -\,|\odot\rangle$$

- In the basis representation:

$$|\uparrow\rangle \;\Longrightarrow\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad |\downarrow\rangle \;\Longrightarrow\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We have:

$$|\otimes\rangle \;\Longrightarrow\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |\odot\rangle \;\Longrightarrow\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \hat{\sigma}_y \;\Longrightarrow\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$