Intermediate Quantum Mechanics
Lecture 24 Notes (4/22/15)

Projection Operators / Bell's Theorem

Projection operators

- **Projection operators** are a very useful tool in making quantum mechanics calculations. We will introduce them here.

- Consider a state in a three dimensional vector space.

$$|\psi\rangle \;=\; \alpha\,|a\rangle \,+\, \beta\,|b\rangle \,+\, \gamma\,|c\rangle$$

- The identity operator: $\hat{\mathbb{P}}_{abc} \;=\; \hat{I} \;=\; |a\rangle\langle a| + |b\rangle\langle b| + |c\rangle\langle c|$ projects out the entire state:

$$\hat{\mathbb{P}}_{abc}\,|\psi\rangle \;=\; |a\rangle\langle a|\psi\rangle \,+\, |b\rangle\langle b|\psi\rangle \,+\, |c\rangle\langle c|\psi\rangle \;=\; \alpha\,|a\rangle \,+\, \beta\,|b\rangle \,+\, \gamma\,|c\rangle \;=\; |\psi\rangle$$

- The projection operator: $\hat{\mathbb{P}}_{ab} \;=\; |a\rangle\langle a| + |b\rangle\langle b|$ projects the state onto the subspace defined by $|a\rangle$ and $|b\rangle$:

$$\hat{\mathbb{P}}_{ab}\,|\psi\rangle \;=\; |a\rangle\langle a|\psi\rangle \,+\, |b\rangle\langle b|\psi\rangle \;=\; \alpha\,|a\rangle \,+\, \beta\,|b\rangle$$

- The projection operator: $\hat{\mathbb{P}}_{a} \;=\; |a\rangle\langle a|$ projects the state onto the subspace defined by $|a\rangle$:

$$\hat{\mathbb{P}}_{a}\,|\psi\rangle \;=\; |a\rangle\langle a|\psi\rangle \;=\; \alpha\,|a\rangle$$

Projection operators for spin-1/2

- A general spin-1/2 state is given by:

$$|\psi\rangle \;=\; \alpha\,|\!\uparrow\rangle \,+\, \beta\,|\!\downarrow\rangle \;\;\longrightarrow\;\; \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

- The projection operator: $\hat{\mathbb{P}}_{\sigma_z\uparrow} \;=\; |\!\uparrow\rangle\langle\uparrow\!|$ projects out the spin-up part of $|\psi\rangle$:

$$\hat{\mathbb{P}}_{\sigma_z\uparrow}\,|\psi\rangle \;=\; |\!\uparrow\rangle\langle\uparrow\!|\psi\rangle \;=\; \alpha\,|\!\uparrow\rangle$$

- Since $\hat{\sigma}_z\,|\!\uparrow\rangle = |\!\uparrow\rangle$ and $\hat{\sigma}_z\,|\!\downarrow\rangle = -\,|\!\downarrow\rangle$, we can write $\hat{\mathbb{P}}_{\sigma_z\uparrow}$ as:

$$\hat{\mathbb{P}}_{\sigma_z\uparrow} \;=\; \frac{\hat{I} + \hat{\sigma}_z}{2} \;\;\longrightarrow\;\; \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

similarly $\hat{\mathbb{P}}_{\sigma_z\downarrow}$ is:

$$\hat{\mathbb{P}}_{\sigma_z\downarrow} \;=\; \frac{\hat{I} - \hat{\sigma}_z}{2} \;\;\longrightarrow\;\; \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

- If $|\psi\rangle$ is expressed in terms of $|\!\rightarrow\rangle$ and $|\!\leftarrow\rangle$, $|\psi\rangle \;=\; \gamma\,|\!\rightarrow\rangle \,+\, \delta\,|\!\leftarrow\rangle$, then clearly:

$$\hat{\mathbb{P}}_{\sigma_x\uparrow} \;=\; \frac{\hat{I} + \hat{\sigma}_x}{2} \;\;\longrightarrow\;\; \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

- It is easy to see that for spin-up in the general $\hat{n}$-direction we have:

$$\hat{\mathbb{P}}_{\sigma_n\uparrow} = \frac{\hat{I} + \hat{\sigma}_n}{2} \longrightarrow \frac{1}{2}\begin{pmatrix} 1 + n_z & n_x - in_y \\ n_x + in_y & 1 - n_z \end{pmatrix}$$

Two alternative ways of determining the probability of measuring an observable

- The main utility of projection operators is that they give us an alternative away of determining the probability of measuring a particular observable for a system that is in a given state.

- Up to now, we used, for example, $\left|\langle n\uparrow | \psi \rangle\right|^2$ to determine the probability of measuring spin-up in the n-direction for an electron in the state $|\psi\rangle$. This requires that we remember that $|n\uparrow\rangle = \cos(\theta/2)\,|\uparrow\rangle + e^{i\phi}\sin(\theta/2)\,|\downarrow\rangle$ or that we determine it.

- An alternative way is to use the expection value of the relevant projection operator. For example, the probability of measuring spin-up in the z-direction is given by:
$$\langle\psi|\hat{\mathbb{P}}_{\sigma_z\uparrow}|\psi\rangle = \langle\psi|\alpha\,|\uparrow\rangle = \Big(\langle\uparrow|\alpha^* + \langle\downarrow|\,\beta^*\Big)\,\alpha\,|\uparrow\rangle = \alpha^*\alpha$$

- Using this method, in order to determine the probability of measuring spin-up in the n-direction for an electron in the state $|\psi\rangle$, we just need to know that

$$\hat{\mathbb{P}}_{\sigma_n\uparrow} = \frac{\hat{I} + \hat{\sigma}_n}{2} \qquad \text{and} \qquad \hat{\sigma}_n = \hat{\sigma}_x n_x + \hat{\sigma}_y n_y + \hat{\sigma}_z n_z \longrightarrow \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

- Projection operators are used in many places. For example, in relativistic quantum mechanics, projections operators for helicity and chirality are used in conjunction with Dirac spinor states.

Bell's Theorem (or Inequality)

- **Bell's Theorem** is a theory related to the logic of set theory. It's importance is that the theorem is violated by quantum mechanics.

- The theorem states that:

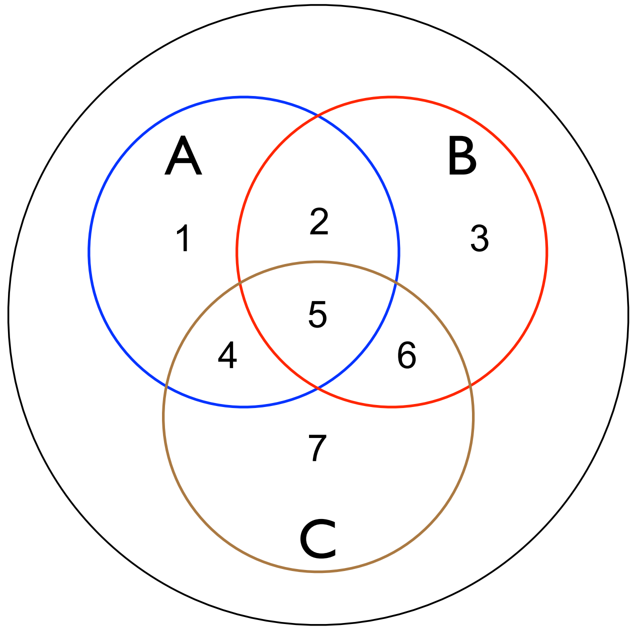
  $(A \text{ and not } B) + (B \text{ and not } C) \geq (A \text{ and not } C) \quad \Rightarrow \quad A\not{B} + B\not{C} \geq A\not{C}$

- The theory is easily proved by referring to a Venn diagram. The Venn diagram below shows the three subsets A, B and C that indicate the regions for which A, B or C is true. Seven distinct regions are labelled 1 to 7. From the diagram, it is easy to see that:

  $A\not{B} = \text{Prob}(1) + \text{Prob}(4) \qquad B\not{C} = \text{Prob}(2) + \text{Prob}(3) \qquad A\not{C} = \text{Prob}(1) + \text{Prob}(2)$

  From which Bell's Theorem easily follows:

  $A\not{B} + B\not{C} \geq A\not{C} \iff \text{Prob}(1) + \text{Prob}(4) + \text{Prob}(2) + \text{Prob}(3) \geq \text{Prob}(1) + \text{Prob}(2)$



Test of Inequality

- We'll now examine whether quantum mechanics violates Bell's Theorem by examining the case of two electrons in the singlet entangled state:

$$\frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big)$$

  - **Proposition** A**:** Electron 1 (Alice's electron) is spin-down in the z-direction.
  - **Proposition** B**:** Electron 1 (Alice's electron) is spin-up in the direction of 135^o with respect to the z-direction in the x-z plane.
  - **Proposition** $\not{B}$**:** Electron 1 (Alice's electron) is **not** spin-up in the direction of 135^o with respect to the z-direction in the x-z plane.
  - **Proposition** C**:** Electron 1 (Alice's electron) is spin-up in the x-direction.
  - **Proposition** $\not{C}$**:** Electron 1 (Alice's electron) is **not** spin-up in the x-direction.

- Since Bob and Alice's electron are in the entangled singlet state, the spins of their two electrons are anti-aligned. So, the propositions $\not{B}$ and $\not{C}$ are equivalent to

  - **Proposition** A**:** Electron 2 (Bob's electron) is spin-up in the z-direction.
  - **Proposition** $\not{B}$**:** Electron 2 (Bob's electron) is spin-up in the direction of 135^o with respect to the z-direction in the x-z plane.
  - **Proposition** $\not{C}$**:** Electron 2 (Bob's electron) is spin-up the x-direction.

- We then have:

  - $\mathbf{A}\not{B}$: Bob's electron spin-up in z and Bob's electron spin-up at 135^o

  - $\mathbf{B}\not{C}$: Bob's electron spin-up at -45^o and Bob's electron spin-up in x.
  - $\mathbf{A}\not{C}$: Bob's electron spin-up in z and Bob's electron spin-up in x.

- In both the case of $\mathbf{A}\not{B}$ and $\mathbf{B}\not{C}$ the two possible spins of Bob's electron are at 135^o with respect to one another. Because of rotational symmetry, these are the same and we then have $\mathbf{A}\not{B} = \mathbf{B}\not{C}$. Bell's inequality then becomes: $2A\not{B} \geq A\not{C}$

The result

- We know right away what the value of $\mathbf{A}\not{C}$ is. If Bob's electron is spin-up in z there there is a 50% probability that it is spin-up in x. So, $\mathbf{A}\not{C} = 0.5$ Just for fun we'll derive this using a projection operator.

$$\mathbf{A}\not{C} \;=\; \langle\uparrow|\hat{\mathbb{P}}_{\hat{\sigma}_x\uparrow}|\uparrow\rangle \;=\; \langle\uparrow|\left(\frac{\hat{I}+\hat{\sigma}_x}{2}\right)|\uparrow\rangle \;=\; \frac{1}{2}\Big(\langle\uparrow|\uparrow\rangle \;+\; \langle\uparrow|\downarrow\rangle\Big) \;=\; 0.5$$

- In order to calculate $\mathbf{A}\not{B}$, we will make use of the projection operator, $\hat{\mathbb{P}}_{\hat{\sigma}_n}$, with $n_x = \sin 135^o = \frac{1}{\sqrt{2}}$, $n_y = 0$ and $n_z = \cos 135^o = -\frac{1}{\sqrt{2}}$.

$$2\mathbf{A}\not{B} \;=\; 2\langle\uparrow|\hat{\mathbb{P}}_{\hat{\sigma}_n}|\uparrow\rangle \;=\; 2\langle\uparrow|\left(\frac{\hat{I}+\hat{\sigma}_n}{2}\right)|\uparrow\rangle$$

$$\longrightarrow \quad \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \;=\; 1-\frac{1}{\sqrt{2}} \;\approx\; 1-0.7 \;=\; 0.3$$

- We've now shown that this situation violates Bell's Theorem. $0.3 \not\geq 0.5$

- In 1964, John Bell proposed his theorem and a means, similar to what we studied above, in which quantum mechanics would violate it. In 1981, Alain Aspect did the definitive experiment that showed that quantum mechanics did indeed violate the theorem. Was it an important experiment? Probably not. Everyone knew what the result was going to be. Quantum mechanics is based on states that are elements of a complex vector space not elements of a set. So, of course it violates the logic of set theory. The (extremely) shocking result would have been if Aspect had found that quantum mechanics didn't violate Bell's Theorem.