Intermediate Quantum Mechanics
Lecture 22 Notes (4/15/15)

Entanglement

The importance of entanglement

- Forty years ago (when I was learning quantum mechanics) no "real" physicist (which doesn't necessarily mean the best) cared about issues of entanglement and questions of what quantum mechanics "means". Textbooks were, by and large, devoid of the subject. Nowadays almost every textbook devotes at least a chapter to the subject. What has changed?
- I can think of the following reasons why entanglement is now considered important. There are likely others
  - Entanglement is closely related to quantum computing that is currently a rather hot topic.
  - There is a possibility (though very unlikely) that quantum mechanics is not a fundament theory but an "effective" theory reflecting something that is deeper. Thinking about what quantum mechanics "means" might then give a hint of what the deeper theory is.
  - A rather radical view taken by some people working in the area of quantum computing is that quantum mechanics is not actually a theory of physics but rather it is a theory of probability involving complex numbers. They would say that physicists just stumbled upon this theory in trying to find a solution to their physics problems. As I mentioned, this is a rather radical point of view.
  - We all enjoy talking about philosophical issues such as what is objective reality or does it even exist. Although most physicists think these discussion are best had over coffee or even better over a few beers.
  - Nevertheless, there are several important and interesting issues involving entanglement that we will discuss in the next four lectures.

Product States

- In the last lecture, we discussed composite systems consisting of more than one particle. Here we will focus on a system consisting of two electrons.
- The vector space of this composite system of two electrons is a 4-dimensional tensor space that is a product of the two 2-dimensional vector spaces of the individual electrons: $V_1 \otimes V_2$.
- The state of each individual electron is given by:

$$|\psi\rangle_1 \;=\; \alpha_1\,|\uparrow\rangle \,+\, \beta_1\,|\downarrow\rangle \qquad\qquad |\psi\rangle_2 \;=\; \alpha_2\,|\uparrow\rangle \,+\, \beta_2\,|\downarrow\rangle$$

- A set of basis states for the 4-dimensional product space is:

$$|\uparrow\uparrow\rangle \qquad\qquad |\uparrow\downarrow\rangle \qquad\qquad |\downarrow\uparrow\rangle \qquad\qquad |\downarrow\downarrow\rangle$$

  where the first arrow represents electron 1 and the second arrow electron 2.

- We can then from a **product state** that is the product of these two individual states.

$$\alpha_1\alpha_2 \left|\uparrow\uparrow\right\rangle + \alpha_1\beta_2 \left|\uparrow\downarrow\right\rangle + \beta_1\alpha_2 \left|\uparrow\downarrow\right\rangle + \beta_1\beta_2 \left|\downarrow\downarrow\right\rangle$$

- The four complex numbers α_1, β_1, α_2, β_2 represent 8 parameters. Two degrees of freedom are removed by the constraints:

$$|\alpha_1|^2 + |\beta_1|^2 = 1 \qquad \text{and} \qquad |\alpha_2|^2 + |\beta_2|^2 = 1$$

  Also, since the overall phase of each individual electron is not physical, we can arbitrarily make α_1 and α_2 real. That leaves 4 degrees of freedom.

- It will be useful to call electron 1 Alice's electron and electron 2 Bob's electron. Then for a product state Alice's and Bob's electrons are independent of one another. (In terms of their states, that is. The electrons will still interact electrically and magnetically with one another.) The fact that they are independent means that Alice can do whatever she want with her electron and it won't effect the state of Bob's electron and vice-versa.

- In particular, both Alice and Bob can find the direction in which their electron is spin-up. They would need to measure an ensemble of electrons all prepared in the same composite state. Then, by making measurements on this ensemble (running their electrons through a Stern-Gelrach device) they will be able to find a particular direction such that when the Stern-Gerlac device is oriented in that direction they always get spin-up. This direction will be given by the direction n such that the expectation value of $\hat{\sigma}_n$ is one: $\langle\hat{\sigma}_n\rangle = 1$. Of course this direction is related to α and β by:

$$\alpha_1 = \cos(\theta_1/2) \qquad \beta_1 = e^{i\phi_1}\sin(\theta_1/2)$$

$$\alpha_2 = \cos(\theta_2/2) \qquad \beta_2 = e^{i\phi_2}\sin(\theta_2/2)$$

Entangled States

- The most general state of a two electron system (considering only the spin part of the state) is

$$\gamma_1 \left|\uparrow\uparrow\right\rangle + \gamma_2 \left|\uparrow\downarrow\right\rangle + \gamma_3 \left|\uparrow\downarrow\right\rangle + \gamma_4 \left|\downarrow\downarrow\right\rangle$$

  The four complex numbers γ_1, γ_2, γ_3, γ_4 represent 8 parameters. We require that the composite state be normalized so that:

$$|\gamma_1|^2 + |\gamma_2|^2 + |\gamma_3|^2 + |\gamma_4|^2 = 1$$

  There is also an arbitrary overall phase that we can use to make one of the γ's real. That leave 6 degrees of freedom. Two more than for the product states. So, for a composite system there are mores states than just the product states. These are states in which the particles are either fully or partially entangled such that the states of the two individual particles are not independent of one another.

- For the two electron spin system, the two maximally entangled states are:

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

  It is easy to see that these are not product states for if they were it would imply the contradiction:

$$\alpha_1\alpha_2 = 0 \qquad \beta_1\beta_2 = 0 \qquad \alpha_1\beta_2 = \frac{1}{\sqrt{2}} \qquad \beta_1\alpha_2 = \frac{1}{\sqrt{2}}$$

- The electrons in either of these two states are maximally entangled. Whatever the spin direction of one electron is the other is guaranteed to be opposite.
- The first state above is called the **singlet** state while the second state is a member of a **triplet** of states.

Spin operators for a composite state

- Label the spin operators for Alice's and Bob's electron $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$. The operator $\hat{\vec{S}}_1$ acts only on Alice's electron and $\hat{\vec{S}}_2$ acts only on Bob's electron. If we consider them as operators in the 4-dimensional product space, then:

$$\hat{\vec{S}}_1 = \frac{\hbar}{2}\hat{\vec{\sigma}} \otimes \hat{I} \qquad \hat{\vec{S}}_2 = \hat{I} \otimes \frac{\hbar}{2}\hat{\vec{\sigma}}$$

  In other words, when acting on Bob's electron, $\hat{\vec{S}}_1$ is just the identity operator and when acting on Alice's electron, $\hat{\vec{S}}_2$ is just the identity operator.
- By convention, we usually define $\hat{\vec{\sigma}}$ and $\hat{\vec{\tau}}$ such that:

$$\hat{\vec{S}}_1 = \frac{\hbar}{2}\hat{\vec{\sigma}} \qquad \hat{\vec{S}}_2 = \frac{\hbar}{2}\hat{\vec{\tau}}$$

  recognizing that $\hat{\vec{\sigma}}$ acts only on Alice's electron and $\hat{\vec{\tau}}$ acts only on Bob's electron.

$$\begin{array}{llll}
\hat{\sigma}_z|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle & \hat{\sigma}_z|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle & \hat{\sigma}_z|\downarrow\uparrow\rangle = -|\downarrow\uparrow\rangle & \hat{\sigma}_z|\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle \\
\hat{\sigma}_x|\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle & \hat{\sigma}_x|\uparrow\downarrow\rangle = |\downarrow\downarrow\rangle & \hat{\sigma}_x|\downarrow\uparrow\rangle = |\uparrow\uparrow\rangle & \hat{\sigma}_x|\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle \\
\hat{\sigma}_y|\uparrow\uparrow\rangle = i|\downarrow\uparrow\rangle & \hat{\sigma}_y|\uparrow\downarrow\rangle = i|\downarrow\downarrow\rangle & \hat{\sigma}_y|\downarrow\uparrow\rangle = -i|\uparrow\uparrow\rangle & \hat{\sigma}_y|\downarrow\downarrow\rangle = -i|\downarrow\downarrow\rangle
\end{array}$$

$$\begin{array}{llll}
\hat{\tau}_z|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle & \hat{\tau}_z|\uparrow\downarrow\rangle = -|\uparrow\downarrow\rangle & \hat{\tau}_z|\downarrow\uparrow\rangle = |\downarrow\uparrow\rangle & \hat{\tau}_z|\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle \\
\hat{\tau}_x|\uparrow\uparrow\rangle = |\uparrow\downarrow\rangle & \hat{\sigma}_x|\uparrow\downarrow\rangle = |\uparrow\uparrow\rangle & \hat{\tau}_x|\downarrow\uparrow\rangle = |\downarrow\downarrow\rangle & \hat{\tau}_x|\downarrow\downarrow\rangle = |\downarrow\uparrow\rangle \\
\hat{\tau}_y|\uparrow\uparrow\rangle = i|\uparrow\downarrow\rangle & \hat{\tau}_y|\uparrow\downarrow\rangle = -i|\uparrow\uparrow\rangle & \hat{\tau}_y|\downarrow\uparrow\rangle = i|\downarrow\downarrow\rangle & \hat{\tau}_y|\downarrow\downarrow\rangle = -i|\downarrow\uparrow\rangle
\end{array}$$

The triplet states

- The triplet state, $\frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\Big)$ is the $m = 0$ member of a $j = 1$ multiplet.

- If we define $\hat{\vec{J}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$, then, as you showed in a homework problem:

$$\hat{J}^2\Big(\,|\!\uparrow\downarrow\rangle + |\!\downarrow\uparrow\rangle\Big) \;=\; 2\hbar^2\Big(\,|\!\uparrow\downarrow\rangle + |\!\downarrow\uparrow\rangle\Big)$$

$$\text{also} \qquad \hat{J}_+\Big(\,|\!\uparrow\downarrow\rangle + |\!\downarrow\uparrow\rangle\Big) \;\propto\; |\!\uparrow\uparrow\rangle \qquad \text{and} \qquad \hat{J}_-\Big(\,|\!\uparrow\downarrow\rangle + |\!\downarrow\uparrow\rangle\Big) \;\propto\; |\!\downarrow\downarrow\rangle$$

- The set of triplet states $|j, m_j\rangle$ is:

$$|1,-1\rangle \;=\; |\!\downarrow\downarrow\rangle \qquad |1,0\rangle \;=\; \frac{1}{\sqrt{2}}\Big(\,|\!\uparrow\downarrow\rangle + |\!\downarrow\uparrow\rangle\Big) \qquad |1,1\rangle \;=\; |\!\uparrow\uparrow\rangle$$

- These states are all symmetric with respect to exchange of the two electrons. Doesn't this violate the Pauli exclusion principle? No. We need to also take into account the spatial part of the state of the two electrons. In addition, to the spin part of the state there is also the wavefunction part. The product of these two must be antisymmetric for two electrons.

$$\big(\psi_\alpha(x_1)\psi_\beta(x_2) \;-\; \psi_\beta(x_1)\psi_\alpha(x_2)\big)\big(\,|\!\uparrow\downarrow\rangle \;+\; |\!\downarrow\uparrow\rangle\big)$$

The singlet state

- The singlet state, $\dfrac{1}{\sqrt{2}}\Big(\,|\!\uparrow\downarrow\rangle - |\!\downarrow\uparrow\rangle\Big)$ is the single $j = 0$ state.

$$|0,0\rangle \;=\; \frac{1}{\sqrt{2}}\Big(\,|\!\uparrow\downarrow\rangle - |\!\downarrow\uparrow\rangle\Big)$$

- If we define $\hat{\vec{J}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$, then, as you showed in a homework problem:

$$\hat{J}^2\Big(\,|\!\uparrow\downarrow\rangle - |\!\downarrow\uparrow\rangle\Big) \;=\; |\emptyset\rangle$$

$$\text{also} \qquad \hat{J}_+\Big(\,|\!\uparrow\downarrow\rangle - |\!\downarrow\uparrow\rangle\Big) \;=\; |\emptyset\rangle \qquad \text{and} \qquad \hat{J}_-\Big(\,|\!\uparrow\downarrow\rangle - |\!\downarrow\uparrow\rangle\Big) \;=\; |\emptyset\rangle$$

- The entire state description including the symmetric wavefunction part is antisymmetric:

$$\big(\psi_\alpha(x_1)\psi_\beta(x_2) \;+\; \psi_\beta(x_1)\psi_\alpha(x_2)\big)\big(\,|\!\uparrow\downarrow\rangle \;-\; |\!\downarrow\uparrow\rangle\big)$$

The spin expectation value for an entangled state

- We will take the singlet state as an example of a maximally entangled state. The spin operator for the first electron (Alice's) is: $\hat{\vec{S}}_1 = (\hbar/2)\,\hat{\vec{\sigma}}$ which acts only on the first electron. The spin operator for the second electron (Bob's) is: $\hat{\vec{S}}_2 = (\hbar/2)\,\hat{\vec{\tau}}$ which acts only on the second electron.

- Let $|S\rangle \equiv \dfrac{1}{\sqrt{2}}\big(\,|\!\uparrow\downarrow\rangle - |\!\downarrow\uparrow\rangle\big)$ then:

$$\langle S|\hat{\sigma}_z\,|S\rangle \;=\; \frac{1}{2}\big(\langle\uparrow\downarrow\!| \;-\; \langle\downarrow\uparrow\!|\big)\hat{\sigma}_z\big(\,|\!\uparrow\downarrow\rangle \;-\; |\!\downarrow\uparrow\rangle\big)$$

$$= \frac{1}{2}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) = \frac{1}{2}\left(\langle\uparrow\downarrow|\downarrow\uparrow\rangle - \langle\downarrow\uparrow|\uparrow\downarrow\rangle\right) = 0$$

$$\langle S|\hat{\tau}_z|S\rangle = \frac{1}{2}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\hat{\tau}_z\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{2}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(-\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \frac{1}{2}\left(-\,\langle\uparrow\downarrow|\downarrow\uparrow\rangle + \langle\downarrow\uparrow|\uparrow\downarrow\rangle\right) = 0$$

$$\langle S|\hat{\sigma}_x|S\rangle = \frac{1}{2}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\hat{\sigma}_x\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$= \frac{1}{2}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\left(|\downarrow\downarrow\rangle - |\uparrow\uparrow\rangle\right) = 0$$

- We could continue but it should be obvious that:

  $$\langle S|\hat{\sigma}_i|S\rangle = \langle S|\hat{\tau}_i|S\rangle = 0$$

  That is, there is no direction of definite spin for either Alice's or Bob's electron. No matter what direction they measure the spin of their electron in they will get 50/50 for spin up and down. This means that we can know everything there is to know about the two electrons, that is, they are in the entangled state $\frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big)$ but we know absolutely nothing about the state of either electron.

- Bob can tell that his electron is entangled with some other electron since he cannot find the spin direction of his electron. If his electron were independent, he would be able to find the direction of its spin.

The Einstein, Rosen, Podolsky (EPR) paradox

- It is easy for Alice and Bob to get their electrons entangled. If they just bring them close together and wait awhile the electrons will become entangled. Because of their spins, the electrons are like small bar magnets. The minimum energy is when the two spins are anti-aligned. This could be either the triplet or the singlet entangled state. We'll show later that of the two the singlet has the lowest energy. So, after some elapsed time the electrons will be in the lowest energy state, the singlet state, $\frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big)$.

- That was the easy part. Now, Alice and Bob can carefully separate their two electrons maintaining the entanglement. This is very difficult since any slight disturbance will cause the two electrons to become at least partially unentangled. But, in principle this can be done.

- After they are separated by a large distance, Alice makes a measurement of her electron by, for example, putting it through a Stern-Gerlach device oriented in the z-direction. The electron will come out either as spin-up in z or spin-down in z with 50/50 probability. Let's say that it comes out as spin-up. Alice's measurement has collapsed the wave function of her electron so that it is definitely in the state with spin-up in z. But, when she did that she instantaneously also collapsed the wavefunction of Bob's electron. If Alice's is spin up in z then Bob's is definitely spin-down in z.

- Is that "spooky"? Well, not yet. Consider the the coin demonstration that we did in class. Give Alice and Bob each a coin that is flipped oppositely from the other. If Alice's coin is heads then Bob's coin is tails and vice versa. Now nobody looks so no one knows if Bob's coin is heads or tails. We can just say that it is 50/50. They separate by a large distance and then Alice looks at her coin and finds that it is heads. We then know instantaneously that Bob's coin is definitely tails. I think most of us would agree that that is not "spooky".

- But there is more to it in a quantum mechanics. As we saw, if Alice measures the spin of her electron in the z direction then Bob's will collapse into a definite spin in the z-direction opposite to the what Alice measured. But what if Alice instead decides to measure the spin of her electron in the x-direction by running it through a Stern-Gerlach device oriented in the x-direction. That will cause Bob's electron to collapse into a state of definite spin in the x direction either spin-up in x or spin-down in x, the opposite of Alice's. But now we see that whether Bob's electron collapses instantaneously into a definite state in the z-direction or into a definite state in the x-direction depends upon what Alice does. Now that's "spooky"

- This is known as the **Einstein-Podolsky-Rosen (EPR) Paradox**. Einstein called it "spooky action at a distance" but he was being (probably intentionally) misleading. It is not spooky action at a distance. It is spooky non locality. There's a big difference.

- Alice spookily instantaneous changes Bob state from a distance (non-locally). But there is nothing that Alice can do that will effect from a distance the results of Bob's observations.

- Let's say that Bob always measures his spin in the z direction. If Alice also measures her spin in the z direction, then half the time she will measure spin-up and half the time spin-down. Bob's state is always the opposite so half the time he will measure spin-up in z and half the time he will measure spin-down in z.

- Now let Alice measure her spin in the x-direction instead. Half the time she will measure spin-up in x and half the time spin-down in x. That means that half the time Bob's electron will be spin-down in x and half the time spin-up in x, the opposite of Alice's. But no matter whether Bob's electron is spin-up or spin-down in x, when he measure the spin of his electron in the z-direction he will find spin-up half the time and spin-down half the time, the same as when Alice measured her spin in the z direction. So, although there is "spooky" non-locality" (Alice can effect Bob's state form a distance), there is no "spooky action at a distance" (Alice cannot effect the outcome of Bob measurement).

- If there were action at a distance, that would mean a breakdown of causality. It would violate special relativity and thus overthrow all of physics. The nature of quantum mechanics although it leads to non-locality does not allow action at a distance. The non-locality is spooky but it doesn't violate any principle of physics.

- Note that after Alice measures her spin in the x-direction and finds spin-up she can then send email to her friend Bob correctly predicting that if Bob measure his spin in x, he will definitely find it to be spin-down. That doesn't violate causality since her email travels at most at the speed of light.

- It's interesting to note that Einstein really didn't like this "spooky" non-locality. For him it was the last straw. The EPR paradox work was in 1935. Although Einstein more than anyone else contributed to the development of many of the basic principles of quantum mechanics, after 1935 he turned his back on quantum mechanics. He spent his last twenty years trying unsuccessfully, to unify classical electromagnetism and gravity. He was divorced from mainstream physics and was pretty much ignored. Of course, being Einstein, he was in some sense ahead of his time. Since around the time of the 1970's, achieving a unification of forces has been one of the principle goals of theoretical physics. In addition, from his work on unifying classical electromagnetism and gravity, with the help of Oskar Klein, the idea of extra dimensions arose. Klein showed that classical electromagnetism and gravity could only be unified in four spatial dimensions. As I'm sure you know, the idea of extra spatial dimensions was made quite popular in recent times by string theory.

Simulating quantum mechanics with a computer

- It is possible to simulate a single electron with a classical computer. The algorithm would be essentially this:
  1) Input two complex numbers α and β that specify the state of the electron.
  2) The computer has a setting that specifies the orientation (θ and ϕ) of a Stern-Gerlach device that is used to measure the electron.
  3) Based on this setting, the computer produces two other complex numbers $\alpha' = \cos(\theta/2)\alpha$ and $\beta' = e^{i\phi}\sin(\theta/2)\beta$.
  4) The computer selects a random number and puts the electron into a spin-up state with probability $|\alpha'|^2$ and into a spin-down state with probability $|\beta'|^2$
  5) The setting on the computer can then be changed and another measurement of the electron made as above.

- Two electrons that are in a composite product state can be simulated by two independent computers. That is Alice can use her computer to simulate her electron and Bob can use his computer to simulate his.

- If the two electrons are in an entangled state, then they cannot be simulated by two independent computers. If Alice and Bob each have computers, then their two computers would have to have an infinitely fast wireless connection in order to simulate the state of the two electrons. That's because what Alice does, instantaneously affects the state of Bob's electron and vice versa. So, the two computers would have to know instantaneously what the other is doing.