Intermediate Quantum Mechanics
Lecture 17 Notes (3/30/15)

Angular Momentum

Ladder operators

- Define the operators $\hat{L}_+$ and $\hat{L}_-$.

$$\hat{L}_+ \equiv \hat{L}_x + i\hat{L}_y \qquad\qquad \hat{L}_- \equiv \hat{L}_x - i\hat{L}_y$$

- Using the commutation relation: $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$, we have:

$$[\hat{L}_+, \hat{L}_z] = -\hbar\hat{L}_+ \qquad\qquad [\hat{L}_-, \hat{L}_z] = \hbar\hat{L}_-$$

- Let $|m\rangle$ be an eigenstate of $\hat{L}_z$ with eigenvalue $m\hbar$. Then:

$$[\hat{L}_+, \hat{L}_z]\,|m\rangle = \left(\hat{L}_+\hat{L}_z - \hat{L}_z\hat{L}_+\right)|m\rangle = -\hbar\hat{L}_+\,|m\rangle$$

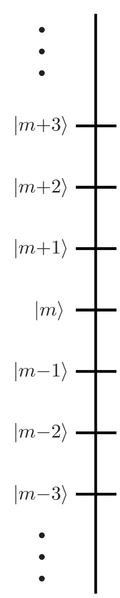
$$\Rightarrow \quad m\hbar\left(\hat{L}_+\,|m\rangle\right) - \hat{L}_z\left(\hat{L}_+\,|m\rangle\right) = -\hbar\left(\hat{L}_+\,|m\rangle\right)$$

$$\Rightarrow \quad \hat{L}_z\left(\hat{L}_+\,|m\rangle\right) = (m+1)\hbar\left(\hat{L}_+\,|m\rangle\right)$$

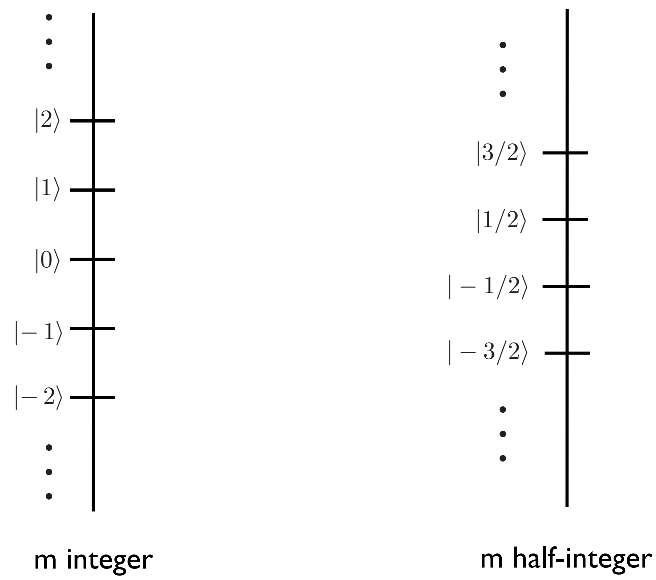
$$\Rightarrow \quad \hat{L}_+\,|m\rangle \text{ is an eigenstate of } \hat{L}_z \text{ with eigenvalue } (m+1)\hbar$$

By a similar argument, $\hat{L}_-\,|m\rangle$ is an eigenstate of $\hat{L}_z$ with eigenvalue $(m-1)\hbar$.

- Starting with the state $|m\rangle$, we can construct the states $|m+1\rangle$, $|m+2\rangle$, $\cdots$ by successively acting with the operator $\hat{L}_+$. Similarly, we can construct the states $|m-1\rangle$, $|m-2\rangle$, $\cdots$ by successively acting with the operator $\hat{L}_-$. This forms a ladder of states.



- By symmetry, these states must be centered about zero. There are only two possibilities for states separated by integers and centered about zero. Either m takes on integer values or m takes on half integer values.



- As we have seen, for orbital angular momentum, the single-valuedness of the wave function requires that m be an integer. In addition to orbital angular momentum, particles can also carry an intrinsic angular momentum called spin. For spin angular momentum there is no wave function and so m can take on either integer or half-integer values. For now we will focus on orbital angular momentum.

Degeneracy

- If the Hamiltonian is spherically symmetric (doesn't depend on θ or ϕ), then we have:

$$[\hat{L}_x, \hat{H}] \;=\; 0 \qquad\qquad [\hat{L}_y, \hat{H}] \;=\; 0 \qquad\qquad [\hat{L}_z, \hat{H}] \;=\; 0$$

  Note that Since, $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}z$ having $[\hat{L}_x, \hat{H}] = [\hat{L}_y, \hat{H}] = 0$ is sufficient to have spherical symmetry since:

$$[\hat{L}_z, \hat{H}] \;=\; -\frac{i}{\hbar}\big[[\hat{L}_x, \hat{L}_y], \hat{H}\big] \;=\; 0$$

- $[\hat{L}_z, \hat{H}] = 0$ means that an eigenstate of $\hat{L}_z$ is also an energy eigenstate.

- For spherical symmetry we also have that:

$$[\hat{L}_+, \hat{H}] \;=\; 0 \qquad\qquad [\hat{L}_-, \hat{H}] \;=\; 0$$

  this means that all of the generated $|m\rangle$ states have the same energy eigenvalue. If $\hat{H}\,|m\rangle = E\,|m\rangle$ then:

$$\hat{H}\,|m+1\rangle \;=\; \hat{H}\hat{L}_+\,|m\rangle \;=\; \hat{L}_+\hat{H}\,|m\rangle \;=\; E\hat{L}_+\,|m\rangle \;=\; E\,|m+1\rangle$$

- Note that in order to have $[\hat{L}_\pm, \hat{H}] = 0$ and therefore for all of the $|m\rangle$ states to be degenerate, there must be full spherical symmetry. Having $[\hat{L}_z, \hat{H}] = 0$ is not sufficient. For example, in the case of an electron in a region of uniform magnetic field in the z direction, there is symmetry with respect to rotation about the z axis but the two spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ have different energies.

The $\hat{L}^2$ operator

- Successive application of the $\hat{L}_+$ operator will continue creating eigenstates of $\hat{L}_z$ with greater eigenvalues until some maximum value of m is reached, m_{max}. Similarly, successive application of the $\hat{L}_-$ operator will continue creating eigenstates of $\hat{L}_z$ with smaller eigenvalues until some minimum value of m is reached, m_{min}. By symmetry $m_{\text{min}} = -m_{\text{max}}$. Let $m_{\text{max}} = l$

$$\hat{L}_z |m_{\text{max}}\rangle = l\hbar |m_{\text{max}}\rangle \qquad \hat{L}_z |m_{\text{min}}\rangle = -l\hbar |m_{\text{min}}\rangle$$

$$\hat{L}_+ |m_{\text{max}}\rangle = |0\rangle \qquad \hat{L}_- |m_{\text{min}}\rangle = |0\rangle$$

- Define the operator: $\hat{L}^2 \equiv \hat{L}^2{}_x + \hat{L}^2{}_y + \hat{L}^2{}_z$

$$\hat{L}^2 = \hat{L}^2{}_z + \left(\hat{L}_x - i\hat{L}_y\right)\left(\hat{L}_x + i\hat{L}_y\right) - i\left[\hat{L}_x, \hat{L}_y\right]$$

$$= \hat{L}^2{}_z + \hbar L_z + \hat{L}_-\hat{L}_+$$

- The eigenvalue of $\hat{L}^2$ is then:

$$\hat{L}^2 |m_{\text{max}}\rangle = (\hat{L}^2{}_z + \hbar L_z + \hat{L}_-\hat{L}_+) |m_{\text{max}}\rangle$$

$$= (l^2 + l)\hbar^2 |m_{\text{max}}\rangle = l(l+1) |m_{\text{max}}\rangle$$

- Since $[\hat{L}^2, \hat{L}_\pm] = 0$, all of the $|m\rangle$ states have the same $\hat{L}^2$ eigenvalue.

Multiplets

- For each value of l, m takes on all integer values between $-l$ and l.

$l = 0$	$m = 0$
$l = 1$	$m = -1,\ 0,\ 1$
$l = 2$	$m = -2,\ -1,\ 0,\ 1\,2$

- For the case of $l = 1$, the states are elements of a three dimensional complex vector space. We can represent them as:

$$|1,1\rangle \rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |1,0\rangle \rightarrow \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |1,-1\rangle \rightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$$

  Note that although these are three dimensional vectors, do not confuse them with arrows in three dimensional physical space.

- For the case of $l = 1$, one choice of the angular momentum operators matrices is:

$$\hat{L}_x \rightarrow \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{L}_y \rightarrow \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \hat{L}_z \rightarrow \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

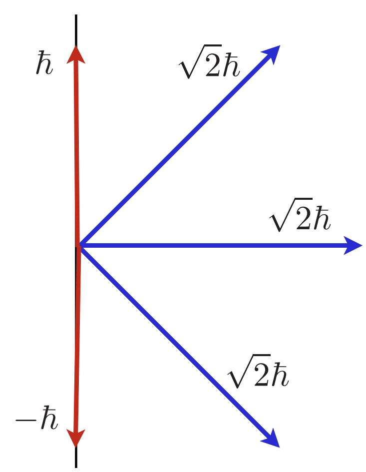
These operators satisfy the $SU(2)$ algebra, $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ink}\hat{L}_k$. They are the generators of a three dimensional representation of $SU(2)$.

The oddness of angular momentum in quantum mechanics

- For a given l, the maximum value of the component of angular momentum measured along any direction, $l\hbar$, is less than the magnitude of the angular momentum, $\sqrt{l(l+1)}\hbar$.

- This means that we cannot determine the direction of the angular momentum. If we could that would mean that we would measure $\sqrt{l(l+1)}\hbar$ for the angular momentum in that direction. It makes sense that we can't do this since if we could determine the direction of the angular momentum, the system would be in a state of definite angular momentum along all three axes, but this is not allowed because the three angular momentum operators do not commute with each other, $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ink}\hat{L}_k$. The state can only have a definite angular momentum in one direction. If we could determine that the angular momentum were definitely in, for example, the z-direction, then the system would have a definite angular momentum in z but it would also have a definite angular momentum in x and y, namely $L_x = L_y = 0$.

- For the case of $l = 1$, the three possibilities $m = 1$, $m = 0$, $m = -1$ are shown in the figure below. the magnitude of the angular momentum is $\sqrt{1(1+1)} = \sqrt{2}$ If the system is in the $m = 1$ state, then the angular momentum arrow points in a direction that makes an angle of 45^o with respect to the z-axis. The component of the annular moment in the z-direction is $\hbar$. The component in the x-y plane is then also $\hbar$ but the individual components in the x and y directions are uncertain.



If the system is in the $m = 0$ state, then the angular momentum arrow points in a direction that makes an angle of 90^o with respect to the z-axis. The component of the annular moment in the z-direction is 0. The component in the x-y plane is then $\sqrt{2}\hbar$ but the individual components in the x and y directions are uncertain.

Aside on solid angle

- In spherical coordinates, the sides of a volume element, dV, are given by $rd\theta$, $r\sin\theta d\phi$ and dr.

$$dV = (rd\theta)(r\sin\theta d\phi)dr$$

- Define the solid angle $d\Omega$ as:

$$d\Omega = \sin\theta d\theta d\phi = (d\cos\theta)d\phi$$

Then $dV = r^2 dr d\Omega$

Spherical harmonics

- We can represent a state $|\psi\rangle$ either in terms of the angular coordinate θ and ϕ or in terms of the angular momentum quantum number l and m.

$$\langle r\theta,\phi\,|\,\psi\,\rangle = \psi(r,\theta,\phi) \qquad \langle r,l,m\,|\,\psi\,\rangle = C_{lm}(r)$$

$$|\psi\rangle = \int_0^\infty\int_0^{2\pi}\int_0^{\pi} \psi(r',\theta',\phi')\,|r',\theta',\phi'\rangle\,{r'}^2 dr' d\Omega'$$

$$|\psi\rangle = \int_0^\infty \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} C_{l'm'}(r')\,|l',m'\rangle dr'$$

- When there is spherical symmetry, the wave function can be factorized as:

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

For the present discussion, we are only interested in the angular dependence. Ignoring the radial dependence we have:

$$|\psi\rangle = \int \psi(\theta',\phi')\,|\theta',\phi'\rangle\,d\Omega' \qquad |\psi\rangle = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} C_{l'm'}\,|l',m'\rangle$$

- We can now find the relationship between $\psi(\theta,\phi)$ and C_{lm}

$$\psi(\theta,\phi) = \langle\theta,\phi\,|\,\psi\,\rangle = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} C_{l'm'}\,\langle\theta,\phi\,|\,l',m'\,\rangle$$

$$C_{lm} = \langle l,m\,|\,\psi\,\rangle = \int \psi(\theta',\phi')\,\langle l,m\,|\,\theta',\phi'\,\rangle\,d\Omega'$$

- The connection between these two representations of $|\psi\rangle$ are given by the spherical harmonics $Y_l^m(\theta,\phi)$.

$$Y_l^m(\theta,\phi) = \langle\theta,\phi\,|\,l',m'\,\rangle \qquad Y_l^{*m}(\theta,\phi) = \langle l',m'\,|\,\theta,\phi\,\rangle$$

$$\psi(\theta,\phi) = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} C_{l'm'}Y_{l'}^{m'}(\theta,\phi) \qquad C_{lm} = \int \psi(\theta',\phi')Y_l^{*m}(\theta',\phi')\,d\Omega'$$

- This is analogous to the connection between the position and momentum representations given by the Fourier transform. Here $\langle \theta, \phi \,|\, l, m \rangle = Y_l^m(\theta, \phi)$ plays the role of $\langle x \,|\, k \rangle = e^{-ikx}$.

- The spherical harmonics satisfy the following orthonormality relations:

$$\int Y_{l'}^{*m'}(\theta, \phi) Y_l^m(\theta, \phi)\, d\Omega \;=\; \delta_{l'l}\delta_{m'm}$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^{*m}(\theta', \phi') Y_l^m(\theta, \phi) \;=\; \delta(\cos\theta' - \cos\theta)\delta(\phi' - \phi)$$

  For a fixed l, we also have:

$$\sum_{m=-l}^{l} Y_l^{*m}(\theta, \phi) Y_l^m(\theta, \phi) \;=\; \frac{2l+1}{4\pi}$$

- The first few spherical harmonics are given below where I have ignored the uninteresting normalization factors:

$$Y_0^0(\theta, \phi) \;\propto\; \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0(\theta, \phi) \;\propto\; \cos\theta \qquad\qquad Y_1^{\pm 1}(\theta, \phi) \;\propto\; \sin\theta\, e^{i\pm\phi}$$

$$Y_2^0(\theta, \phi) \;\propto\; 3\cos^2\theta - 1 \qquad\qquad Y_2^{\pm 1}(\theta, \phi) \;\propto\; \sin\theta\cos\theta\, e^{i\pm\phi} \qquad\qquad Y_2^{\pm 2}(\theta, \phi) \;\propto\; \sin^2\theta\; e^{i2\pm\phi}$$

A very nice picture of the spherical harmonics for the cases of $l = 0, 1, 2, 3$ is shown here in a figure from the Wikipedia article on spherical harmonics.