Intermediate Quantum Mechanics
Lecture 16 Notes (3/25/15)

Rotations and Angular Momentum

Generator of Rotations

- We can find the generators of rotation transformations similarly to the way we found the generators of space translations in the last lecture. Here we will work in spherical rather than Cartesian coordinates and we will focus on rotations about the z-axis. In terms of spherical coordinates, rotations about the z-axis correspond to translation in the azimuthal angle ϕ.

$$\hat{U}_{R_z}(\epsilon)\,|\psi\rangle \;=\; |\psi'\rangle$$

$$\langle r,\theta,\phi|\hat{U}_{R_z}(\epsilon)\,|\psi\rangle \;=\; \langle r,\theta,\phi\,|\,\psi'\,\rangle \;=\; \psi'(r,\theta,\phi) \;=\; \psi(r,\theta,\phi-\epsilon)$$

$$=\; \psi(r,\theta,\phi) - \epsilon\,\frac{\partial}{\partial\phi}\,\psi(r,\theta,\phi) \;=\; \left(1 \,-\, \epsilon\,\frac{\partial}{\partial\phi}\right)\psi(r,\theta,\phi) \;=\; \left[1 \,-\, \frac{i\epsilon}{\hbar}\left(-i\hbar\,\frac{\partial}{\partial\phi}\right)\right]\psi(r,\theta,\phi)$$

$$\Rightarrow \quad \langle r,\theta,\phi|\hat{U}_{R_z}(\epsilon)\,|\psi\rangle \;=\; \left[1 \,-\, \frac{i\epsilon}{\hbar}\left(-i\hbar\,\frac{\partial}{\partial\phi}\right)\right]\psi(r,\theta,\phi)$$

$$\Rightarrow \quad \hat{U}_{R_z}(\epsilon)\,|\psi\rangle \;=\; \hat{I} \,-\, \frac{i\epsilon}{\hbar}\,\hat{L}_z \qquad \text{with} \qquad \langle r,\theta,\phi|\hat{L}_z\,|\psi\rangle \;=\; -i\hbar\,\frac{\partial}{\partial\phi}\,\psi(r,\theta,\phi)$$

- A finite rotation about the z-axis is then given by:

$$\hat{U}_{R_z}(\phi_0) \;=\; e^{i\phi_0\hat{L}_z/\hbar}$$

The Angular Momentum Operator

- We'll now work out the form of the generator of rotations about the z-axis, in the position representation using Cartesian coordinates.

- In spherical coordinates:

$$x \;=\; r\cos\phi\sin\theta \qquad\qquad y \;=\; r\sin\phi\sin\theta \qquad\qquad z \;=\; r\cos\theta$$

  Let $\rho \;=\; r\sin\theta$, then:

$$x \;=\; \rho\cos\phi \qquad\qquad y \;=\; \rho\sin\phi \qquad\qquad z \;=\; r\cos\theta$$

- We then have:

$$\frac{\partial}{\partial\phi} \;=\; \frac{\partial x}{\partial\phi}\,\frac{\partial}{\partial x} \,+\, \frac{\partial y}{\partial\phi}\,\frac{\partial}{\partial y} \,+\, \frac{\partial z}{\partial\phi}\,\frac{\partial}{\partial z}$$

$$=\; (-\rho\sin\phi)\,\frac{\partial}{\partial x} \,+\, (\rho\cos\phi)\,\frac{\partial}{\partial y} \;=\; x\,\frac{\partial}{\partial y} \,-\, y\frac{\partial}{\partial x}$$

$$\Rightarrow \quad \langle x,y,z|\hat{L}_z|\psi\rangle \;=\; (-i\hbar)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \;=\; x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)$$

$$\Rightarrow \quad \hat{L}_z \;=\; \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

As can be seen by comparison with the classical dynamic variable, this is the angular momentum operator corresponding to angular momentum along the z-axis.

- It's easy to guess what $\hat{L}_x$ and $\hat{L}_y$ are:

$$\hat{L}_x \;=\; \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \qquad \hat{L}_y \;=\; \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad \hat{L}_z \;=\; \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

- The generators of rotations about the x, y and z axes are, respectively, the x, y and z angular momentum operators.

Eigenvalues of $\hat{L}_z$

- We will now find the eigenstates and eigenvalues of the $\hat{L}_z$ operator. Let $|m\rangle$ be an eigenstate of $\hat{L}_z$ with eigenvalue $m\hbar$.

$$\hat{L}_z|m\rangle \;=\; m\hbar|m\rangle$$

- Let's work in the position representation using spherical coordinates.

$$\langle r,\theta,\phi|\hat{L}_z|m\rangle \;=\; m\hbar\langle r,\theta,\phi\,|\,m\rangle \;=\; m\hbar\,\psi_m(r,\theta,\phi)$$

$$\Rightarrow \quad -i\hbar\frac{\partial}{\partial\phi}\psi_m(r,\theta,\phi) \;=\; m\hbar\,\psi_m(r,\theta,\phi)$$

- This is a differential equation in ϕ that is independent of r and θ. That means that we can write $\psi_m(r,\theta,\phi)$ as a product of a function of r and θ and a separate function of ϕ.

$$\psi_m(r,\theta,\phi) \;=\; F(r,\theta)\Phi_m(\phi)$$

- The differential equation above now involves only $\Phi_m(\phi)$.

$$-i\hbar\frac{\partial}{\partial\phi}\Phi_m(\phi) \;=\; m\hbar\,\Phi_m(\phi)$$

This has the solution:

$$\Phi_m(\phi) \;=\; Ae^{im\phi}$$

Quantization

- The wavefunction must have a specific value at each point in space. That means the wavefunction must be a single-valued function of its arguments. Since ϕ is a cyclic coordinate, we must have:

$$\Phi_m(\phi+2\pi) \;=\; \Phi_m(\phi) \qquad \Rightarrow \qquad e^{im(\phi+2\pi)} \;=\; e^{im\phi} \qquad \Rightarrow \qquad e^{im2\pi} \;=\; 1$$

$$\Rightarrow \qquad m2\pi \;=\; n2\pi \qquad \text{where } n \text{ is an integer}$$

$$\Rightarrow \qquad m \text{ is an integer} \qquad m \;=\; \cdots,-2,-1,0,1,2,\cdots$$

- The eigenvalues of $\hat{L}_z$ are quantized. They can only take on integer values of $\hbar$.

Angular Momentum Commutation Relations

- Using the Cartesian expressions for the $\hat{L}$ operators:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \qquad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

  and the canonical commutation relations:

$$[x_i, x_j] = 0 \qquad [p_i, p_j] = 0 \qquad [x_i, p_j] = i\hbar\delta_{ij}$$

  it is straightforward to determine the commutation relations of the $\hat{L}$'s.

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

- We can summarize these as:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k \qquad \text{where } \epsilon_{ijk} \text{ is the Levi-Civita symbol}$$

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \qquad \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \qquad \text{all other } \epsilon_{ijk} = 0$$

- These commutation relations completely determine the properties of the $\hat{L}$'s. They form what is called a Lie algebra. The rotation transformations that they generate form a Lie group.

Group Theory

- A mathematical **group** is a collection of elements with a defined operation that satisfies the following properties.
  - The group is closed under the operation. $E_1 \cdot E_2 = E_3$
  - The operation is associative. $E_1 \cdot (E_2 \cdot E_3) = (E_1 \cdot E_2) \cdot E_3$
  - There is an identity element. $E \cdot I = E$
  - Every element has an inverse. $E \cdot E^{-1} = I$ with E^{-1} an element of the group

  Note there is not a requirement that the operation be commutative.

- The set of all possible rotations in three dimensional physical space with the operation defined as the product of rotations forms a group.
  - Closure: $\hat{U}_{R_\alpha}(\theta_\alpha)\hat{U}_{R_\beta}(\theta_\beta)$ is a rotation
  - Associativity: $\hat{U}_{R_\alpha}(\theta_\alpha)\Big(\hat{U}_{R_\beta}(\theta_\beta)\hat{U}_{R_\gamma}(\theta_\gamma)\Big) = \Big(\hat{U}_{R_\alpha}(\theta_\alpha)\hat{U}_{R_\beta}(\theta_\beta)\Big)\hat{U}_{R_\gamma}(\theta_\gamma)$
  - Identity element: $\hat{U}_{R_i}(\theta_i)\hat{I} = \hat{U}_{R_i}(\theta_i)$
  - Inverse: $\hat{U}_{R_i}(\theta_i)\hat{U}_{R_i}(-\theta_i) = \hat{I}$

- A **Lie group** is a group whose elements are differentiable functions of their parameter(s). The group of all rotations is a Lie group since $\hat{U}_{R_i}(\theta_i) = e^{-i\theta_i\hat{L}_i/\hbar}$ is a differentiable function of its parameter θ_i.

- The structure of the group of rotations is determined by the commutation relations of its generators

$$[\hat{L}_i, \hat{L}_j] \; = \; i\hbar\epsilon_{ijk}\hat{L}_k$$

  This is called its Lie algebra.

$SU(2)$

- The group of all unitary, 2-dimensional operators is called $U(2)$. It is the set of unitary transformations that are generated by the set of all 2-dimensional Hermitian operators. Since a two-dimensional Hermitian operator has four independent parameters,

$$\begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix}$$

  there are four of these. A possible choice is: $\hat{I}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$.

- The operator $\hat{I}$ generates the transformations:

$$\hat{U}_I(\theta) \; = \; e^{i\theta\hat{I}} \; = \; e^{i\theta}\hat{I}$$

  As we have discussed before we are not interested in these transformations since they simply multiply a state by an overall complex phase that has no physics effect.

$$\hat{U}_I(\theta)\,|\psi\rangle \; = \; e^{-i\theta}\,|\psi\rangle$$

- We therefore consider a more restricted group that consists of all transformations generated by the set of all traceless 2-dimensional Hermitian matrices. This group is called **SU(2)**.

- The three σ matrices satisfy the following algebra:

$$[\hat{\sigma}_i, \hat{\sigma}_j] \; = \; 2i\epsilon_{ijk}\hat{\sigma}_k$$

  This is not exactly the Lie algebra of rotations.

- We define $\hat{S}_i \; = \; \hbar/2\,\hat{\sigma}_i$. These then satisfy the $SU(2)$ Lie algebra:

$$[\hat{S}_i, \hat{S}_j] \; = \; i\hbar\epsilon_{ijk}\hat{S}_k$$

- The operators: $\hbar/2\,\hat{\sigma}_x$, $\hbar/2\,\hat{\sigma}_y$, $\hbar/2\,\hat{\sigma}_z$ are the generators of the 2-dimensional manifestation of SU(2). As we will see, there are also higher dimensional (3, 4, 5, $\cdots$) manifestations of SU(2).