Intermediate Quantum Mechanics
Lecture 13 Notes (3/4/15)
Simple Harmonic Oscillator II

Matrix elements of $\hat{a}$ and $\hat{a}^\dagger$

- We'll now calculate the constants that result when the raising and lowering operators act on states

$$\hat{a}\,|E_n\rangle \;=\; c_n\,|E_n\rangle \qquad\Rightarrow\qquad \langle E_n|\hat{a}^\dagger \;=\; c_n^*$$

$$\langle E_n|\hat{a}^\dagger\hat{a}\,|E_n\rangle \;=\; n \qquad\Rightarrow\qquad |c_n|^2 \;=\; n$$

We can absorb an overall phase into the eigenstates and make c_n real.
Then $c_n = \sqrt{n}$.

- We have then

$$\hat{a}\,|E_n\rangle \;=\; \sqrt{n}\,|E_{n-1}\rangle \qquad \hat{a}^\dagger\,|E_{n-1}\rangle \;=\; \sqrt{n}\,|E_n\rangle \quad \text{or} \quad \hat{a}^\dagger\,|E_n\rangle \;=\; \sqrt{n+1}\,|E_{n+1}\rangle$$

$$\langle E_{n-1}|\hat{a}\,|E_n\rangle \;=\; \langle E_n|\hat{a}^\dagger\,|E_{n-1}\rangle \;=\; \sqrt{n} \qquad\qquad \langle E_{n+1}|\hat{a}^\dagger\,|E_n\rangle \;=\; \sqrt{n+1}$$

Ground eigenstate wave function

- We will now find the ground state wave function.

$$\hat{a}\,|E_0\rangle \;=\; |\emptyset\rangle \quad\Rightarrow\quad \left(\hat{x'}+i\hat{p'}\right)|E_0\rangle \;=\; |\emptyset\rangle \quad\Rightarrow\quad \langle x|\left(\hat{x'}+i\hat{p'}\right)|E_0\rangle \;=\; \langle x\,|\,\emptyset\,\rangle$$

This gives a simple first order differential equation:

$$\left(x' + \frac{d}{dx'}\right)u_0(x') \;=\; 0$$

- It's easy to see that the solution to this equation is:

$$u_0(x') \;=\; A_0' e^{-{x'}^2/2}$$

$$\text{Using} \qquad x' \;=\; \sqrt{\frac{m\omega}{\hbar}}\,x, \qquad \text{we have:} \qquad u_0(x) \;=\; A_0 e^{-m\omega x^2/2\hbar}$$

First executed state

- We now solve for the first excited state $u_1(x)$.

$$\hat{a}^\dagger\,|E_0\rangle \;=\; |E_1\rangle \quad\Rightarrow\quad \left(\hat{x'}-i\hat{p'}\right)|E_0\rangle \;=\; |E_1\rangle \quad\Rightarrow\quad \langle x|\left(\hat{x'}-i\hat{p'}\right)|E_0\rangle \;=\; \langle x\,|\,E_1\,\rangle$$

This gives the first order differential equation:

$$\left(x' - \frac{d}{dx'}\right)u_0(x') \;=\; u_1(x')$$

- From the differential equation for $u_0(x')$ above we have that:

$$\frac{d}{dx'}\,u_0(x') \;=\; -\,x'u_0(x')$$

which gives: $\quad u_1(x') \;=\; 2x'u_0(x') \;=\; A_1'x'e^{-x'^2/2}$

Using $\quad x' \;=\; \sqrt{\frac{m\omega}{\hbar}}x, \qquad$ we have: $\quad u_1(x) \;=\; A_1xe^{-m\omega x^2/2\hbar}$

- We can now get $u_2(x')$ by solving

$$\left(x' \;-\; \frac{d}{dx'}\right)u_1(x') \;=\; u_2(x')$$

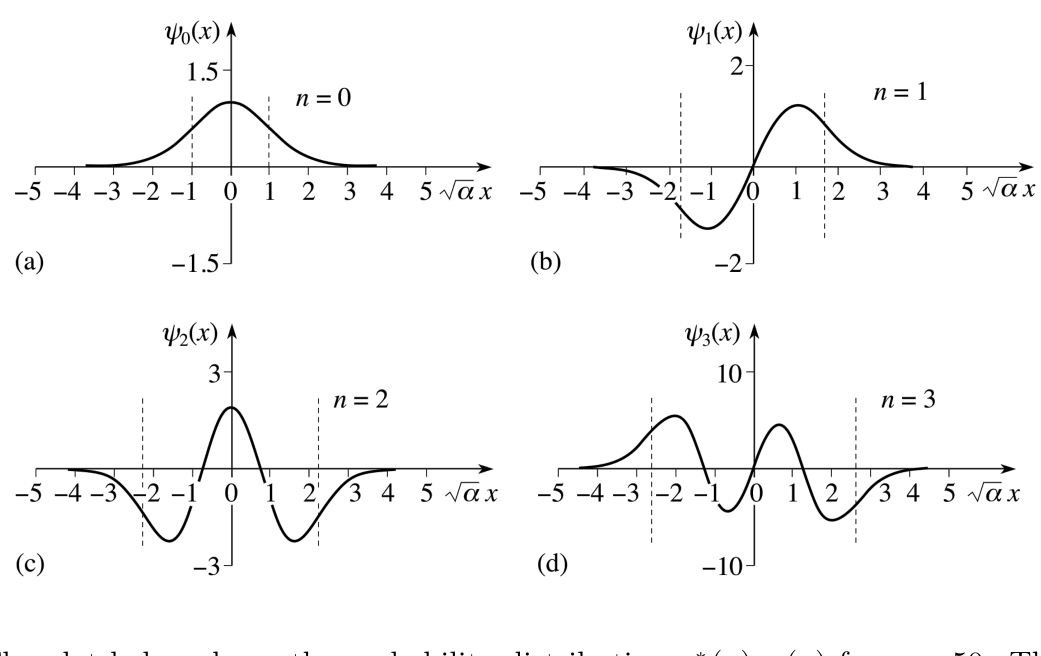
and continue on getting all of the $u_n(x')$'s. We would find:

$$u_n(x') \;=\; H_n(x')e^{-x'^2/2}$$

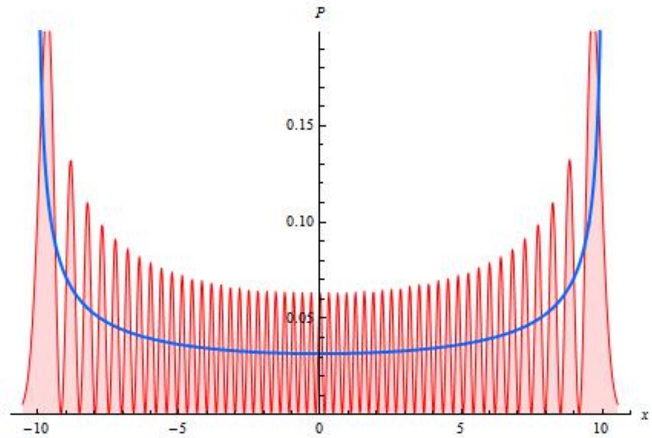
where $H_n(x')$ is the Hermite polynomial of order n. The even n Hermite polynomials have only even power terms in x' and the odd n Hermite polynomials have only odd power terms in x'. If you are interested, you can find a listing of the Hermite polynomials on wikipedia or in almost any quantum mechanics text book.

Plots of wave functions and probability distributions

- The plots below show the $n = 0$, $n = 1$, $n = 2$ and $n = 3$ wave functions.



- The plot below shows the probability distribution $u_n^*(x)u_n(x)$ for $n = 50$. The blue curve is the classical expectation. It is the position probability distribution that you would get if you randomly took photographs of a mass on the end of a spring. For a very high energy states as $n \to \infty$, the position distribution of the quantum harmonic oscillator approaches the classical case.



Calculating SHO matrix elements

- Using:

$$\hat{x'} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \text{and} \qquad \hat{p'} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

  and the orthonormality of the energy eigenstates, we can evaluate the matrix elements of any powers of $\hat{x'}$ or $\hat{p'}$ between energy eigenstates.

- For example,

$$\langle E_m|\hat{x'}\,|E_n\rangle = \frac{1}{\sqrt{2}}\langle E_m|\left(\hat{a}^\dagger + \hat{a}\right)|E_n\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{n+1}\,\langle E_m\,|\,E_{n+1}\,\rangle + \sqrt{n}\,\langle E_m\,|\,E_{n-1}\,\rangle\right)$$

$$= \sqrt{\frac{n+1}{2}}\,\delta_{m,n+1} + \sqrt{\frac{n}{2}}\,\delta_{m,n-1}$$