Intermediate Quantum Mechanics
Lecture 10 Notes (2/23/15)
Spin in a Uniform Magnetic Field

Spin in a Uniform Magnetic Field

- In the last lecture, we derived the expression for the time development of a system in terms of the energy eigenvalues. We also derived the Ehrenfest Theorem that relates the rate of change of the average value of an observable to the average value of the commutation of the corresponding operator with the Hamiltonian.

$$|\psi(t)\rangle \;=\; \sum_i a_i(0)e^{-iE_it/\hbar}\,|E_i\rangle$$

$$\frac{d}{dt}\,\langle\psi(t)|\hat{A}\,|\psi(t)\rangle \;=\; -\frac{i}{\hbar}\,\langle\psi(t)|\,[\hat{A},\hat{H}]\,|\psi(t)\rangle$$

  In this lecture, we'll apply these results to the simple 2-dimensional vector state system of electron spin.

The Hamiltonian for a free electron at rest

- First consider the case of an electron at rest in free space. Take the case that the electron is in the state with spin-up in the x-direction at time $t = 0$.

$$|\psi(0)\rangle \;=\; |\rightarrow\rangle \;=\; \frac{1}{\sqrt{2}}\big(\,|\uparrow\rangle + |\downarrow\rangle\big)$$

- The energy of the electron is just its mass energy, $E = mc^2$. Both $|\uparrow\rangle$ and $|\downarrow\rangle$ are energy eigenstates with energy mc^2. We then have:

$$|\psi(t)\rangle \;=\; \frac{1}{\sqrt{2}}\left(e^{-imc^2t/\hbar}\,|\uparrow\rangle \,+\, e^{-imc^2t/\hbar}\,|\downarrow\rangle\right) \;=\; \frac{1}{\sqrt{2}}\,e^{-imc^2t/\hbar}\big(\,|\uparrow\rangle + |\downarrow\rangle\big)$$

  The state of the system is changing by just an overall time dependent complex phase.

$$|\psi(t)\rangle \;=\; e^{-imc^2t/\hbar}\,|\psi(0)\rangle$$

  As we've seen before, this means that $|\psi(t)\rangle$ is physically indistinguishable from $|\psi(0)\rangle$ so we can just ignore the overall $e^{-imc^2t/\hbar}$ factor. It has no effect.

The Hamiltonian for an electron in a uniform magnetic field

- Recall that in classical electromagnetism, the energy of a magnetic dipole in a region of magnetic field is: $E \;=\; -\vec{\mu}\cdot\vec{B}$. It turns out soothing similar is true in quantum mechanics:

$$\hat{H} \;=\; -\left(\hat{\mu}_xB_x \,+\, \hat{\mu}_yB_y \,+\, \hat{\mu}_zB_z\right)$$

  The difference is that the magnetic dipole is not an arrow but an operator.

- The relations between the magnetic dipole operators ($\hat{\mu}$), the spin operators ($\hat{s}$) and the sigma operators ($\hat{\sigma}$) are:

$$\hat{\mu}_x = -\frac{e}{m}\,\hat{s}_x = -\frac{e\hbar}{2m}\,\hat{\sigma}_x \qquad \hat{\mu}_x = -\frac{e}{m}\,\hat{s}_y = -\frac{e\hbar}{2m}\,\hat{\sigma}_y \qquad \hat{\mu}_z = -\frac{e}{m}\,\hat{s}_z = -\frac{e\hbar}{2m}\,\hat{\sigma}_z$$

  The minus sign is due to the fact that the charge of the electron is $-e$.

- Let's consider a region of uniform magnetic field. That means that everywhere in the resign the magnetic field points in the same direction and has the same magnitude. We might as well choose the z-direction to be in the direction of the magnetic field. Then:

$$\hat{H} = -\hat{\mu}_z B = \frac{eB}{m}\,\hat{s}_z = \frac{eB\hbar}{2m}\,\hat{\sigma}_z$$

- Since $\hat{H}$ is proportional to $\hat{\sigma}_z$, the energy eigenstates are the eigenstates of $\hat{\sigma}_z$, spin-up and spin-down in the z direction $|\uparrow\rangle$ and $|\downarrow\rangle$ with eigenvalues $\pm\frac{eB\hbar}{2m}$.

$$\hat{H}\,|\uparrow\rangle = \frac{eB\hbar}{2m}\,\hat{\sigma}_z\,|\uparrow\rangle = \frac{eB\hbar}{2m}\,|\uparrow\rangle \qquad \hat{H}\,|\downarrow\rangle = \frac{eB\hbar}{2m}\,\hat{\sigma}_z\,|\downarrow\rangle = -\frac{eB\hbar}{2m}\,|\downarrow\rangle$$

- Define the cyclotron frequency $\omega \equiv eB/m$. Then:

$$\hat{H}\,|\uparrow\rangle = \frac{\omega}{2}\,|\uparrow\rangle \qquad \hat{H}\,|\downarrow\rangle = -\frac{\omega}{2}\,|\downarrow\rangle$$

- Take the case that $|\psi(0)\rangle = |\rightarrow\rangle$. Then:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega t/2}\,|\uparrow\rangle + e^{i\omega t/2}\,|\downarrow\rangle\right)$$

  Now the relative phase of the $|\uparrow\rangle$ and $|\downarrow\rangle$ components is changing and this has a definite effect on the state.

Precession

- We can take a common factor of $e^{-i\omega t/2}$ out of the above expression for $|\psi(t)\rangle$:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\,e^{-i\omega t/2}\left(|\uparrow\rangle + e^{i\omega t}\,|\downarrow\rangle\right)$$

- At $t = 0$, the $|\uparrow\rangle$ and $|\downarrow\rangle$ components are in phase:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) = |\rightarrow\rangle$$

  When $\omega t = \pi/2$, the $|\uparrow\rangle$ and $|\downarrow\rangle$ components are $\pi/2$ out of phase:

$$|\psi(\pi/2\omega)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i\,|\downarrow\rangle\right) = |\otimes\rangle$$

When $\omega t = \pi$, the $|\uparrow\rangle$ and $|\downarrow\rangle$ components are π out of phase:

$$|\psi(\pi/\omega)\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle \;-\; |\downarrow\rangle\Big) \;=\; |\leftarrow\rangle$$

After one cycle when $\omega t = 2\pi$, the $|\uparrow\rangle$ and $|\downarrow\rangle$ components are π back in phase:

$$|\psi(2\pi/\omega)\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle \;+\; |\downarrow\rangle\Big) \;=\; |\rightarrow\rangle$$

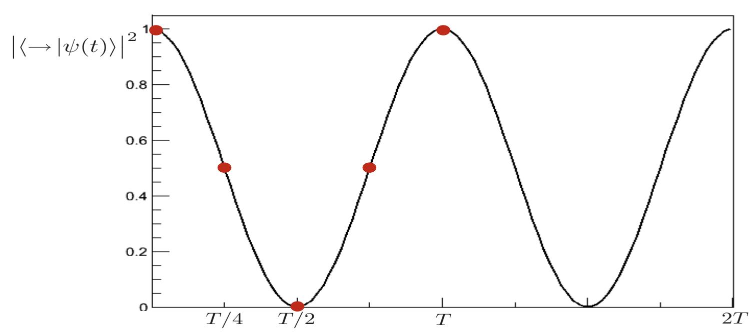
The state of the system precesses around the z-axis with angular frequency $\omega = eB/m$. This is just what a magnetic dipole in a uniform magnetic filed does classically. But, as you should expect by now, the situation in quantum mechanics is very different.

Time Dependent Probability of measuring the $|\rightarrow\rangle$ State

- Let's find the probability to measure spin-up in the x-direction as a function of time. By one of the rules (postulates) of quantum mechanics, this is given by the square of the $|\rightarrow\rangle$ component of $|\psi(t)\rangle$:

$$\big|\langle\rightarrow|\,\psi(t)\,\rangle\big|^2 \;=\; \left|\frac{1}{\sqrt{2}}\Big(\langle\uparrow| \,+\, \langle\downarrow|\Big)\frac{1}{\sqrt{2}}\Big(e^{-i\omega t/2}\,|\uparrow\rangle \;+\; e^{i\omega t/2}\,|\downarrow\rangle\Big)\right|$$

$$=\; \left|\frac{1}{2}\Big(e^{-i\omega t/2} \,+\, e^{i\omega t/2}\Big)\right|^2 \;=\; \cos^2(\omega t/2)$$



- At $t = 0$, the electron is in the state $|\rightarrow\rangle$ and the probability to measure spin-up in the x-direction is 100%. After a quarter of a period, the electron is in the state with spin-up in the y=direction, $|\otimes\rangle$. The probability to measure spin-up in x is 50%. After half a period, the electron is in the state with spin-down in the x-direction, $|\leftarrow\rangle$ and the probability of measuring spin-up in x is zero.

- This is very different from the classical case of spin precession. There the electron is initially in the state of spin-up in the x-direction. But, as soon as it starts to precess it is no longer spin-up in the x-direction and won't be again until after one period when it has precessed completely around.

Ehrenfest's Theorem applied to the case of spin in a magnetic field

- We will now use the Ehrenfest Theorem to show that the average value of the spin agrees with what we expect for the component of the spin in the classical case.

- The rate of chafe of the average value of spin in the x-direction is given by:

$$\frac{d}{dt}\langle\psi(t)|\hat{\sigma}_x|\psi(t)\rangle \;=\; -\frac{i}{\hbar}\langle\psi(t)|[\hat{\sigma}_x,\hat{H}]\,|\psi(t)\rangle \;=\; -\frac{ieB}{2m}\langle\psi(t)|[\hat{\sigma}_x,\hat{\sigma}_z]\,|\psi(t)\rangle$$

  Using $\quad [\hat{\sigma}_i,\hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k \quad \Rightarrow \quad [\hat{\sigma}_x,\hat{\sigma}_z] = -2i\hat{\sigma}_y \quad$ and $\quad \omega = eB/m$:

$$\frac{d}{dt}\langle\psi(t)|\hat{\sigma}_x|\psi(t)\rangle \;=\; -\omega\,\langle\psi(t)|\hat{\sigma}_y|\psi(t)\rangle$$

- Similarly,

$$\frac{d}{dt}\langle\psi(t)|\hat{\sigma}_y|\psi(t)\rangle \;=\; -\frac{i}{\hbar}\langle\psi(t)|[\hat{\sigma}_y,\hat{H}]\,|\psi(t)\rangle \;=\; -\frac{ieB}{2m}\langle\psi(t)|[\hat{\sigma}_y,\hat{\sigma}_z]\,|\psi(t)\rangle \;=\; \omega\,\langle\psi(t)|\hat{\sigma}_x|\psi(t)\rangle$$

- Using somewhat poor but useful notation:

$$\dot{\sigma}_x \;=\; -\omega\sigma_y \qquad \text{and} \qquad \dot{\sigma}_y \;=\; \omega\sigma_x$$

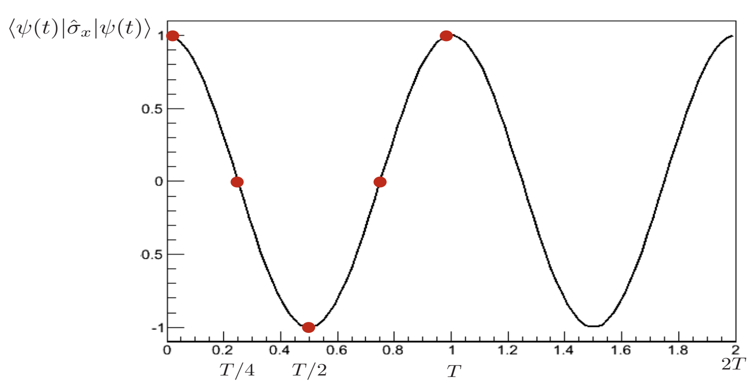
  We then have:

$$\ddot{\sigma}_x \;=\; -\omega^2\sigma_x \qquad \text{and} \qquad \ddot{\sigma}_y \;=\; -\omega^2\sigma_y$$

  These have solutions:

$$\langle\psi(t)|\hat{\sigma}_x|\psi(t)\rangle \;=\; \cos\omega t \qquad \text{and} \qquad \langle\psi(t)|\hat{\sigma}_y|\psi(t)\rangle \;=\; \sin\omega t$$

  These have the same dependence on time as the respective components of the classical spin. A single electron behaves very different than a classical particle. If we prepare may electron in identical states and make identical measurements on these electrons, the average of the measurements agree with a the classical case.



The General Case

- We consider a more general case in the initial state of the electron is spin-up in the n-direction.

$$|\psi(0)\rangle \;=\; |n\!\uparrow\rangle \;=\; e^{-i\phi/2}\cos(\theta/2)\,|\uparrow\rangle \;+\; e^{i\phi/2}\sin(\theta/2)\,|\downarrow\rangle$$

Then: $$|\psi(t)\rangle \;=\; e^{-i\phi/2}\cos(\theta/2)e^{-iwt/2}\,|\uparrow\rangle \;+\; e^{i\phi/2}\sin(\theta/2)e^{iwt/2}\,|\downarrow\rangle$$

- You can work this out as an exercise if you like. You will find:

$$\langle\psi(t)|\hat{\sigma}_x\,|\psi(t)\rangle \;=\; \sin\theta\cos(\omega t+\phi)$$

$$\langle\psi(t)|\hat{\sigma}_y\,|\psi(t)\rangle \;=\; \sin\theta\sin(\omega t+\phi)$$

$$\langle\psi(t)|\hat{\sigma}_z\,|\psi(t)\rangle \;=\; \cos\theta$$

- Note that $\langle\psi(t)|\hat{\sigma}_z\,|\psi(t)\rangle$ is independent of time since:

$$\frac{d}{dt}\,\langle\psi(t)|\hat{\sigma}_z\,|\psi(t)\rangle \;=\; -\frac{ieB}{2m}\,\langle\psi(t)|[\hat{\sigma}_z,\hat{\sigma}_z]\,|\psi(t)\rangle \;=\; 0$$

Quantum Mechanics and Reversibility

- As we have seen, the time development of the state of a system is given by :

$$|\psi(t)\rangle \;=\; \hat{U}(t)\,|\psi(0)\rangle$$

This is reversible: $$|\psi(0)\rangle \;=\; \hat{U}(-t)\,|\psi(t)\rangle \;=\; \hat{U}^\dagger(t)\,|\psi(t)\rangle$$

- Quantum mechanics like classical physics is reversible. Information is conserved. If we know the state of a system at time t and the Hamiltonian is fixed (does't change with time). we know what the state of the system will be for all times in the future and what it was for all times in the past.

- If we make a measurement on the system, we necessarily disturb it (we leave it in an eigenstate of the operator whose observable we measured). This completely changes the evolution of the system with time. It is not reversible since the outcome of the measurement was determined probabilistically. It is not reversible unless we "undo" the measurement. That means that the measurement device must be included as part of the system. We must also then describe the measuring device by quantum mechanics.

- The conundrum with measurement in quantum mechanics is that we normally consider the measuring device to be a macroscopic, classical device, for example, the complicated electromagnet with all of tis power supplies and electronics. We are necessarily forced to do this since it is essentially impossible to work out the quantum mechanical description of such a complicated device.

- In a few lectures from now, we will discuss the famous “two-slit” experiment. We’ll see then a measuring device that is a simple quantum mechanical device. This will lead to some very interesting insights into the issue of measurement in quantum mechanics.