"The Polhode Rolls without Slipping on the Herpolhode Lying in the Invariable Plane"

The motion of a rigid body upon which no external forces (or torques) act is a standard problem in intermediate classical mechanics. Because the center of mass is unaccelerated, the problem may be treated in a reference frame in which it is and remains at the origin, so the fundamental degrees of freedom are those describing the rotation of the body from a fixed initial position, represented by an orthogonal matrix A. There are only three independent degrees of freedom here, and the velocities can be taken as the angular velocity vector ω. But the configuration space is the group manifold of the rotation group, a three dimensional space but not a flat one, and the motion can be quite complex.

In this simulation, the angular velocity vector, expressed in body fixed coordinates, ω_{body}, has its motion determined by Euler's equations. The time evolution of the matrix A is determined by $dA_{im}/dt = \epsilon_{ijk}\omega_k A_{jm}$. Vectors in the lab (inertial) frame are related to those in the body fixed frame by $V_{body} = AV_{lab}$.

The problem is somewhat simplified by the conservation of two quantities, the kinetic energy $T = \omega \cdot I \cdot \omega/2$, and the angular momentum $L = I \cdot \omega$, where I is the moment of inertia tensor. The angular momentum is a vector with components conserved only in the inertial lab frame. Because of the energy conservation, ω at any instant is confined to an ellipsoid given by the quadratic constraint. This ellipsoid is fixed in the body coordinate system (where the moment of inertia tensor is fixed), and is called the inertia ellipsiod. On the other hand, as $T = L\omega/2$ and T and L are conserved in laboratory coordinates, one component of ω is fixed in laboratory coordinates, confining ω_{lab} to the invariable plane. The locus of points mapped on the inerta ellipsoid by $\omega_{body}(t)$ is called the polhode, and the locus of points mapped by $\omega_{lab}(t)$ on the invariable plane is called the herpolhode. As the momentary position of ω is on the axis of rotation at the moment, and points on this axis are momentarily at rest, the polhode rolls without slipping on the herpolhode.

This program generates an object with three independant principal moments of inertia, and shows the body, its inertia ellipsoid, and the polhode and herpolhode. It integrates the equations of motion, and can show the configuration at various times.