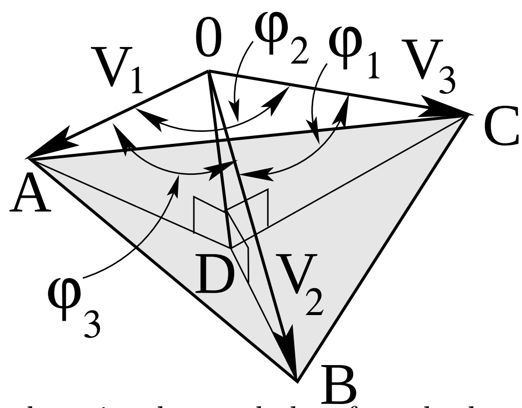
The Sum of Angles Between Three Vectors

Joel A. Shapiro

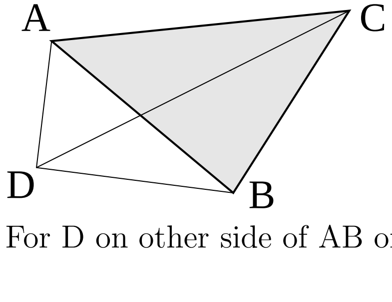
Given three vectors in $\mathbb{R}^3$, $\vec{V}_j$, the sum of the angles between pairs of them is $\leq 360°$, and $= 360°$ only if the three vectors are coplanar.

Proof: Let ϕ_1 be the angle between $\vec{V}_2$ and $\vec{V}_3$, and similarly $\phi_2 = \angle(\vec{V}_1, \vec{V}_3)$, $\phi_3 = \angle(\vec{V}_1, \vec{V}_2)$. Clearly the angles between these vectors does not depend on their length, so let them all have the same length V. Place the vectors with their tails at the origin, and call the points at their heads A, B and C. ABC forms a triangle in a plane (shaded). Drop a perpendicular from the origin 0 to this plane at D, and call its length d.

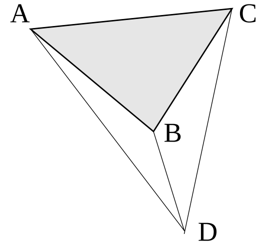


Note that the length AD is $\sqrt{V^2 - d^2}$, and so are the lengths BD and CD. Call this length L. Thus 0AB and DAB are both isosceles triangles, and therefore the length $(AB) = 2L \sin \frac{\angle ADB}{2}$, but it is also $2V \sin \frac{\phi_3}{2}$. But unless the three vectors are coplanar, $d > 0$ and $V > L$, so $\sin \frac{\phi_3}{2} < \sin \frac{\angle ADB}{2}$. As sine is monotone increasing for these angles, which are all $\in [0, \pi/2]$, we have $\phi_3 < \angle ADB$. Similarly $\phi_2 < \angle ADC$ and $\phi_1 < \angle BDC$, but $\angle ADB + \angle ADC + \angle BDC = 2\pi$, so $\phi_1 + \phi_2 + \phi_3 < 2\pi$, unless the three vectors are coplanar.

The diagram assumed that D lies within the triangle, but the same argument holds even if it lies outside, for example on the other side of AB but on the same side of AC and BC, in which case $\angle ADB = \angle ADC + \angle BDC \leq \pi$, or on the other side of both AB and BC, in which case $\angle ADC = \angle ADB + \angle BDC \leq \pi$, so in either case we now have $\angle ADB + \angle ADC + \angle BDC \leq 2\pi$, and $\phi_1 + \phi_2 + \phi_3 < 2\pi$, unless the three vectors are coplanar.



For D on other side of AB only



For D on other side of AB and BC