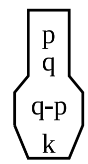
Some notes trying to understand "$q-p$" notation (P 117,121)

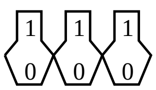
The simple roots are α^i, $i = 1 \ldots m$, and any positive root is $\phi_k = \sum_j k_j \alpha^j$ for $k_j \in \mathbb{N}$, (with at least one non-zero). The Cartan matrix is defined as

$$A_{ji} = \frac{2\alpha^i \cdot \alpha^j}{(\alpha^i)^2}.$$

Georgi represents each root with a box containing $q^i - p^i$ for each simple root α^i. I will use a more complex figure, a composite of bottles for each simple root, Each bottle contains, in its widest part, the same $q^i - p^i$ Georgi puts in his box, but also the k_i at the bottom and the p^i above the q^i in the neck. Of course this is redundant information, but it helps in tracing the diagram.

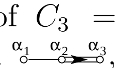


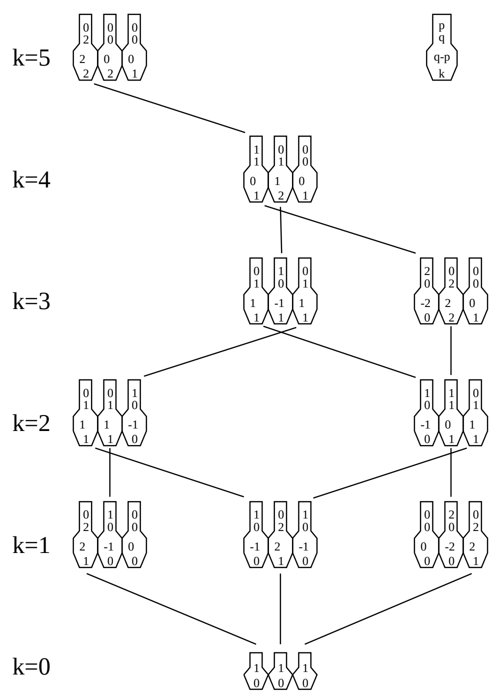
In the diagram, each positive root will consist of a string of touching bottles, one for each simple root. The diagram starts with a different symbol,



, where all the $k_i = 0$ and all the p^i are 1, because each simple root can act only once ($E_{2\vec{\alpha}}$ is not a root). There will be m of these truncated bottles for a rank m algebra.

The simple roots correspond to a $k = \sum k_i = 1$, one level up from the figure representing the Cartan subalgebra. For each of these roots, E_{α^i}, we have $q_j = -2\delta_{ij}$ and $p_i = 0$, because $E_{-\alpha}$, $\alpha \cdot H$, and E_α are the only states parallel to α, and no other simple root can lower a simple root, because $[E_\alpha, E_\beta] = 0$ for $\alpha \neq \beta$. The $q-p$ value for the i'th bottle in the j'th simple root is A_{ji} because only $k_j \neq 0$. The p values are then determined.

Consider the example of $C_3 = \mathrm{Sp}(6)$, with Dynkin diagram , with $(\alpha^1)^2 = (\alpha^2)^2 = 1$, $(\alpha^3)^2 = 2$, and $A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.



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The $q-p$ values for the j'th root are just the entries in the j'th row of the Cartan matrix above. Thus we have the $k=1$ level in the diagram shown.

The general formula for the number of raisings p and the number of lowerings q that E_α can give to a weight μ state, is

$$2\frac{\vec{\alpha}\cdot\vec{\mu}}{\vec{\alpha}^{\,2}} = q - p.$$

When E_{α^j} acts on a state μ with q^i and p^i values for α^i (with $p^j \neq 0$), the state $E^j_\alpha\,|\mu\rangle$ has $k_i \to k_i + \delta_{ij}$, $\mu \to \mu + \alpha^j$ and $q^i - p^i \to q^i - p^i + A_{ji}$. So for each bottle with $p > 0$, there is a path up to one for E^j_α acting on it with the the k_i and $q^i - p^i$ values incremented, and with its q^j one more than the q^j it came from.

A root may have more than one path leading up to it, with q^i's determined as just mentioned, and with $q^i = 0$ if there is no E_{α^i} leading up to it. With all the $q^i - p^i$ and q^i so determined, the p^i are also determined, and so we can continue generating or connecting new roots until all the highest k nodes have all $p^i = 0$.