Hyperspherical Coordinates (in N dimensions)

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In N dimensional Euclidean space, with cartesian coordinates x_j, $j = 1, \cdots, N$, we could define hyperspherical coordinates as follows.

First define a radius $r = \rho_1 \geq 0$ with $\rho_1^2 = \sum_{j=1}^{N} x_j^2$. The subspace of constant $\rho_1 > 0$ is an $N-1$ dimensional hypersphere S^{N-1}. The intersection of this hypersphere with the hyperplane of constant $x_1 = \rho_1 \cos\theta_1$ is an $N-2$ dimensional hypersphere with radius

$$\rho_2 = \rho_1 \sin\theta_1, \qquad \rho_2^2 = \sum_{j=2}^{N} x_j^2,$$

for a fixed $\theta_1 \in [0, \pi]$. The intersection of this hypersphere with the hyperplane of fixed $x_2 = \rho_2 \cos\theta_2$ is an $N-3$ dimensional hypersphere of radius $\rho_3 = \rho_2 \sin\theta_2$, and so on, so

$$\rho_j^2 = r^2 \prod_{k=1}^{j-1} \sin^2\theta_k = \sum_{k=j}^{N} x_k^2, \qquad x_j = \rho_j \cos\theta_j = r \left(\prod_{k=1}^{N-1} \sin\theta_k \right) \cos\theta_k,$$

for $j = 1, \ldots, N-1$. The final $x_N = \rho_{N-1} \sin\theta_{N-1}$. The first $N-2$ θ's range from 0 to π, ($\theta_k \in [0, \pi]$ for $k = 1, \ldots, N-2$), because only their $\cos\theta_k = x_k/\rho_k$ is determined, but θ_{N-1} determines both x_{N-1} and x_N so $\theta_{N-1} \in [0, 2\pi]$ with $\theta_{N-1} = 0$ identified with $\theta_{N-1} = 2\pi$.

Our normal 3-dimensional coordinates have $\rho_1 = r$, $x_1 = z$, $\theta_1 = \theta$ (for physicists), $\theta_2 = \phi$, $x_2 = x = r \sin\theta \cos\phi$, $x_3 = y = r \sin\theta \sin\phi$.

The differential $dx_j = d\rho_j \cos\theta_j - \rho_j \sin\theta_j d\theta_j$, except for $dx_N = d\rho_{N-1} \sin\theta_{N-1} + \rho_{N-1} \cos\theta_{N-1} d\theta_{N-1}$. Thus

$$\begin{aligned} (dx_N)^2 &= (d\rho_{N-1})^2 \sin^2\theta_{N-1} \\ &\quad + 2\rho_{N-1} d\rho_{N-1} d\theta_{N-1} \sin\theta_{N-1} \cos\theta_{N-1} + \rho_{N-1}^2 \cos^2\theta_{N-1} (d\theta_{N-1})^2 \\ (dx_j)^2 &= (d\rho_j)^2 \cos^2\theta_j - 2\rho_j d\rho_j d\theta_j \cos\theta_j \sin\theta_j + \rho_j^2 \sin^2\theta_j (d\theta_j)^2. \end{aligned}$$

Thus we see

$$\begin{aligned}
(dx_{N-1})^2 + (dx_N)^2 &= (d\rho_{N-1})^2 + \rho_{N-1}^2 \, (d\theta_{N-1})^2 \\
&= (d\rho_{N-2})^2 \sin^2 \theta_{N-2} \\
&\qquad +2\rho_{N-2} \, d\rho_{N-2} \, d\theta_{N-2} \sin \theta_{N-2} \cos \theta_{N-2} \\
&\qquad +\rho_{N-2}^2 \, \cos^2 \theta_{N-2} \, (d\theta_{N-2})^2 + \rho_{N-1}^2 \, (d\theta_{N-1})^2
\end{aligned}$$

where in the second expression I used (with $j = N{-}2$)

$$d\rho_{j+1} = d\rho_j \sin \theta_j + \rho_j \cos \theta_j d\theta_j .$$

Adding this to

$$\begin{aligned}
(dx_{N-2})^2 &= (d\rho_{N-2})^2 \cos^2 \theta_{N-2} \\
&\qquad -2\rho_{N-2} d\rho_{N-2} d\theta_{N-2} \cos \theta_{N-2} \sin \theta_{N-2} + \rho_{N-2}^2 \sin^2 \theta_{N-2} (d\theta_{N-2})^2
\end{aligned}$$

we see that

$$\begin{aligned}
&(dx_{N-2})^2 + (dx_{N-1})^2 + (dx_N)^2 \\
&= (d\rho_{N-2})^2 + \rho_{N-1}^2 \, (d\theta_{N-1})^2 + \rho_{N-2}^2 \, (d\theta_{N-2})^2
\end{aligned}$$

This points the way to inductively find $(ds)^2$. I claim

$$\sum_{k=j}^{N} (dx_k)^2 = (d\rho_j)^2 + \sum_{k=j}^{N-1} \rho_k^2 (d\theta_k)^2 \qquad \text{for} \quad j = 1, \ldots, N-1. \tag{1}$$

We have just shown it is true for $j = N{-}1$ and $j = N{-}2$. Suppose it is true for j. Noting that

$$(d\rho_j)^2 = (d\rho_{j-1})^2 \sin^2 \theta_{j-1} + \sin(2\theta_{j-1}) \, \rho_{j-1} \, d\rho_{j-1} \, d\theta_{j-1} + \rho_{j-1}^2 \cos^2 \theta_{j-1} \, (d\theta_{j-1})^2,$$

$$\begin{aligned}
\sum_{k=j-1}^{N} (dx_k)^2 &= (dx_{j-1})^2 + (d\rho_j)^2 + \sum_{k=j}^{N-1} \rho_k^2 (d\theta_k)^2 \\
&= (d\rho_{j-1})^2 \cos^2 \theta_{j-1} - \sin(2\theta_{j-1}) \, \rho_{j-1} \, d\rho_{j-1} \, d\theta_{j-1} \\
&\qquad +\rho_{j-1}^2 \sin^2 \theta_{j-1} \, (d\theta_{j-1})^2 + (d\rho_{j-1})^2 \, \sin^2 \theta_{j-1} \\
&\qquad + \sin(2\theta_{j-1}) \rho_{j-1} d\rho_{j-1} d\theta_{j-1} + \rho_{j-1}^2 \cos^2 \theta_{j-1} (d\theta_{j-1})^2 \\
&\qquad + \sum_{k=j}^{N-1} \rho_k^2 (d\theta_k)^2 \\
&= (d\rho_{j-1})^2 + \sum_{k=j-1}^{N-1} \rho_k^2 (d\theta_k)^2,
\end{aligned}$$

so by induction we have shown Eq. (1).

So $(ds)^2 = (dr)^2 + \sum_{k=1}^{N-1} \rho_k^2 (d\theta_k)^2$, and the hypervolume element is

$$d^N x = \prod_{k=1}^{N} dx_k = r^{N-1}\, dr \prod_{k=1}^{N-1} \left(\sin^{N-k-1} \theta_k \; d\theta_k \right).$$