Group Invariant Metric

We have defined a group invariant **measure** $\mu(\nu)$ which gives the "volume density" of parameter space $d\nu^1 d\nu^2 ... d\nu^n$. We now turn to defining a **metric**, a bilinear form

$$(ds)^2 = g_{ab} d\nu^a d\nu^b$$

which gives the square of the distance between a group element $U(\nu)$ and a nearby one $U(\nu + d\nu)$. We would like to have this distance have the property that if both group elements are multiplied from the left by the same group element G, the distance between the two images is the same as the original distance. That is also true if G acts from the right.

In the neighborhood of the identity we may take the metric to be given by the Killing form. That is, the distance between $\mathbb{1}$ and $e^{i\nu^a L_a}$ is given by $(ds)^2 = \beta(\nu^a L_a, \nu^b L_b)$, where β is the Killing form. We saw that for compact semisimple Lie algebras we could always take $\beta(L_a, L_b) = \delta_{ab}$, which we do here, even though many physicist's generators are off by a factor of two from this. So $(ds)^2 = \sum_a \nu^a \nu^a$, and $g_{ab}(\nu = 0) = \delta_{ab}$.

Now let's consider the distance between $U(\nu)$ and $U(\nu + d\nu)$ by noting that it is to be the same as distance between $U(\nu+d\nu)U^{-1}(\nu)$ and $\mathbb{1}$. Writing

$$U(\nu + d\nu)U^{-1}(\nu) = e^{i(\nu+d\nu)^a L_a} e^{-i\nu^a L_a} = e^{i\rho^a L_a},$$

or

$$e^{i(\nu+d\nu)^a L_a} = e^{i\rho^a L_a} e^{i\nu^a L_a}.$$

Differentiating this with respect to ν^a is of the form we considered before,

$$\begin{aligned}
\frac{\partial}{\partial \nu^a} e^{i(\nu+d\nu)^c L_c} &= \sum_b \frac{\partial \rho^b}{\partial \nu^a} \frac{\partial}{\partial \rho^b} e^{i\rho^c L_c} e^{i\nu^c L_c} = i \sum_b \frac{\partial \rho^b}{\partial \nu^a} L_b e^{i\nu^c L_c} \\
&= \int_0^1 d\alpha \, e^{i\alpha\nu^c L_c} i L_a e^{i(1-\alpha)\nu^c L_c} = i \left[\frac{1 - e^{-i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)} \right]_{ab} L_b e^{i\nu^c L_c},
\end{aligned}$$

where $\mathcal{S}$ is the adjoint representation of the Lie algebra and the matrix in brackets is to have zero eigenvalues of $\nu^c \mathcal{S}(L_c)$ cancelled top and bottom, giving 1 on these eigenvectors. This is the form we saw before for the relationship between

$$\frac{\partial}{\partial \nu^a} \quad \text{and} \quad E_b.$$

Again we conclude

$$d\rho^b = \sum_a d\nu^a \left[\frac{1 - e^{-i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right]_{ab},$$

and the distance is given by

$$ds^2 = \sum_b \rho^b \rho^b = \sum_b \left[\frac{1 - e^{-i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right]_{ab} \left[\frac{1 - e^{-i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right]_{cb} d\nu^a d\nu^c.$$

Now $\mathcal{S}$ is hermetian and imaginary, so antisymmetric, and we can take the transpose

$$\left[\frac{1 - e^{-i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right]_{cb} = -\left[\frac{1 - e^{i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right]_{bc}$$

and the metric is

$$\begin{aligned} (ds)^2 &= -d\nu^a \left(\left[\frac{1 - e^{-i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right] \left[\frac{1 - e^{i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right]\right)_{ac} d\nu^c \\ &= \left[\frac{e^{-i\nu^c \mathcal{S}(L_c)} + e^{-i\nu^c \mathcal{S}(L_c)} - 2}{(i\nu^c \mathcal{S}(L_c))^2}\right]_{ab} d\nu^a d\nu^b \end{aligned}$$

so

$$g_{ab}(\nu) = \left[\frac{e^{i\nu^c \mathcal{S}(L_c)} + e^{-i\nu^c \mathcal{S}(L_c)} - 2}{(i\nu^c \mathcal{S}(L_c))^2}\right]_{ab} = \left(\left[\frac{e^{\frac{1}{2}i\nu^c \mathcal{S}(L_c)} - e^{-\frac{1}{2}i\nu^c \mathcal{S}(L_c)}}{(i\nu^c \mathcal{S}(L_c))}\right]^2\right)_{ab}.$$

In general in Riemannian geometry the measure $\mu(\nu) = \sqrt{g(\nu)} := \sqrt{\det g_{ab}(\nu)}$, so we see in this case that

$$\mu(\nu) = \sqrt{g(\nu)} = \det \left[\frac{e^{\frac{1}{2}i\nu^c \mathcal{S}(L_c)} - e^{-\frac{1}{2}i\nu^c \mathcal{S}(L_c)}}{i\nu^c \mathcal{S}(L_c)}\right].$$