

## Lecture 26

Dec. 2, 2013

## Non-Abelian Gauge Theory, part 2

Copyright©2012 by Joel A. Shapiro

Last time, after polishing off Goldstone's theorem on spontaneously broken global symmetries, we began our discussion of making a theory with local symmetry. We used a latticized model to make some of the concepts more concrete. That is, we considered *matter* fields  $\phi_j(x^\mu)$  defined only on a four-dimensional lattice of points,  $x^\mu = a n^\mu$ ,  $n^\mu \in \mathbb{Z}$ , where  $a$  is a lattice spacing. The matter fields are taken to transform linearly under the symmetry group transformation  $G$ ,  $\phi_j(x^\mu) \rightarrow M_{jk}(G(x^\mu)) \phi_k(x^\mu)$ , with  $M_{jk}(G)$  a representation of the group. We saw that even if the lagrangian is invariant under this group transformation when  $G$  is the same at every point in spacetime, it will not be if we ask for local invariance, where  $G$  can vary with  $x^\mu$ . Because the kinetic terms in the lagrangian  $(\partial_\mu \phi_j)^2$  depend on how the matter fields change from one point to another, we need to carefully define what it means to subtract the value at one point from that at another. On the lattice these kinetic terms involve the coupling of nearest neighbor points. To compare symmetry-noninvariant quantities at different points, we need to *parallel transport* the field at  $x$  to  $x + \Delta$  before subtracting it from  $\phi(x + \Delta)$ . Ordinarily we assume the basis vectors are the same from point to point, and parallel transport just keeps components unchanged. But with local definitions the basis vectors at one point are a symmetry transformation of those at another. Thus parallel transport is a symmetry transformation, so the rule for parallel transport is to apply a group element  $G_L$ , so the  $\Delta\phi$  we need for the kinetic energy term is

$$\Delta\phi_j = \phi_j(x + \Delta/2) - M_{jk}(G_L(x)) \phi_k(x - \Delta/2),$$

where  $x$  is at the middle of the link  $L$  running from  $x - \Delta/2$  to  $x + \Delta/2$ .

In the continuum limit, assuming the local symmetry varies smoothly, the group element to parallel transport by one lattice spacing in the  $\mu$  direction will be close to the identity, so we may write  $G_L = e^{iag\mathcal{A}_\mu}$ , where  $\mathcal{A}_\mu$  is a generator of the Lie algebra ( $a\mathcal{A}_\mu$  is an infinitesimal transformation as  $a \rightarrow 0$ ). Taking  $L_b$  to be a basis of generators of the Lie algebra (or group),  $\mathcal{A}_\mu = A_\mu^{(b)} L_b$ , and  $M(G_L) = e^{iagM(\mathcal{A}_\mu)} \approx \mathbb{I} + iagM(\mathcal{A}_\mu) = \mathbb{I} + iagA_\mu^{(b)} M(L_b)$ .

Then we may make a locally invariant theory by replacing  $\partial_\mu \phi_j$  by the

covariant derivative

$$(D_\mu \phi)_j := \partial_\mu \phi_j - ig A_\mu^{(b)} M_{jk} (L_b) \phi_k,$$

and our theory will be invariant under gauge transformation  $\lambda(x) = \sum_b \lambda^{(b)}(x) L_b$  with

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = e^{i\lambda^{(b)}(x) M(L_b)} \phi(x) \\ \mathcal{A}_\mu(x) &\rightarrow \mathcal{A}'_\mu(x) = A_\mu^{(b)}(x) L_b = e^{i\lambda} \left( \mathcal{A}_\mu + \frac{i}{g} \partial_\mu \right) e^{-i\lambda} \end{aligned}$$

Having introduced new degrees of freedom  $\mathcal{A}_\mu(x)$  (or  $A_\mu^{(b)}(x)$ ), we need a kinetic term for them in the Lagrangian. We turn to that now.

This is just an introduction to the main part of the lecture, which is “Gauge Theory on a Lattice”, starting at page 8 today.

[Reminder: You might want to read “Lightning Review of Groups”.]