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This discussion has been somewhat abstract, so it might be well to give some examples. We will consider

- the pendulum
- the two-dimensional isotropic harmonic oscillator
- the three dimensional isotropic anharmonic oscillator

The Pendulum

The simple pendulum is a mass connected by a fixed length massless rod to a frictionless joint, which we take to be at the origin, hanging in a uniform gravitational field. The generalized coordinates may be taken to be the angle θ which the rod makes with the **downward** vertical, and the azimuthal angle ϕ. If ℓ is the length of the rod, $U = -mg\ell\cos\theta$, and as shown in section **??** (**??**) or section **??**, the kinetic energy is $T = \frac{1}{2}m\ell^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right)$. So the lagrangian,

$$L = \frac{1}{2}m\ell^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) + mg\ell\cos\theta$$

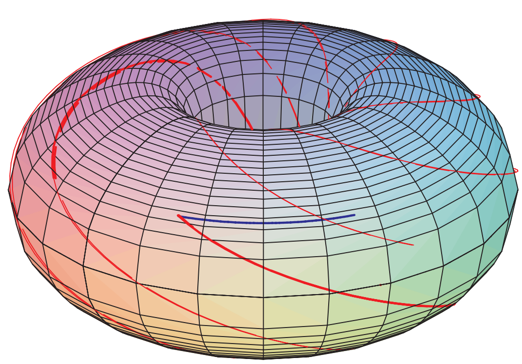
is time independent and has an ignorable coordinate ϕ, so $p_\phi = m\ell^2\sin^2\theta\dot\phi$ is conserved, and so is H. As $p_\theta = m\ell^2\dot\theta$, the Hamiltonian is

$$H = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mg\ell\cos\theta.$$

In the four-dimensional phase space one coordinate, p_ϕ, is fixed, and the equation $H(\theta,\phi,p_\theta) = E$ gives a two-dimensional surface in the three-dimensional space which remains. Let us draw this in cylindrical coordinates with radial coordinate θ and z coordinate p_θ. Thus the motion will be restricted to the invariant torus

♣♠ ⇒ *is this where that is defined???*

shown below. The generators $F_2 = p_\phi$ and $F_1 = H$ generate motions along the torus as shown, with p_ϕ generating changes in ϕ, leaving θ and p_θ fixed. Thus a point moves as on the blue path shown, looking



like a line of latitude. The change in ϕ generated by $g^{(0,t_2)}$ is just t_2, so we may take $\phi = phi_2$ of the last section. H generates the dynamical motion of the system,

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}, \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\theta}{m\ell^2 \sin^2\theta},$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos\theta}{m\ell^2 \sin^3\theta} - mg\ell\sin\theta.$$

This is shown by the red path, which goes around the bottom, through the hole in the donut, up the top, and back, but not quite to the same point as it started. Ignoring ϕ, this is periodic motion in θ with a period T_θ, so $g^{(T_\theta,0)}(\eta_0)$ is a point at the same latitude as η_0. This $t \in [0, T_\theta]$ part of the trajectory is shown as the thick red curve. There is some $\bar{t}_2$ which, together with $\bar{t}_1 = T_\theta$, will cause $g^{\vec{\bar{t}}}$ to map each point on the torus back to itself.

Thus $\vec{e}_1 = (T_\theta, \bar{t}_2)$ and $\vec{e}_2 = (0, 2\pi)$ constitute the unit vectors of the lattice of $\vec{t}$ values which leave the points unchanged. The trajectory generated by H does not close after one or a few T_θ. It could be continued indefinitely, and as in general there is no relation among the frequencies ($\bar{t}_2/2\pi$ is not rational, in general), the trajectory will not close, but will fill the surface of the torus. If we wait long enough, the system will sample every region of the torus.

The 2-D isotropic harmonic oscillator

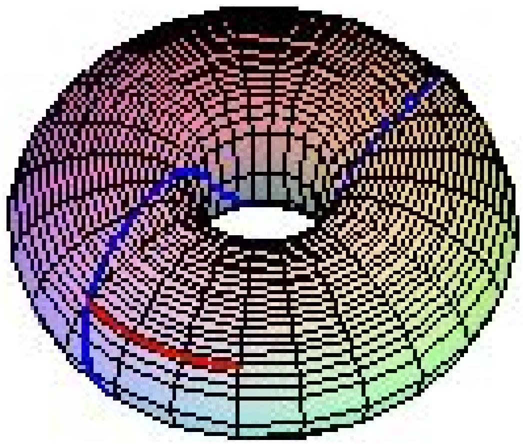
A different result occurs for the two dimensional zero-length isotropic oscillator,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2.$$

While this separates in cartesian coordinates, from which we easily see that the orbit closes because the two periods are the same, we will look instead at polar coordinates, where we have a conserved Hamiltonian

$$F_1 = H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{2}kr^2,$$

and conserved momentum p_ϕ conjugate to the ignorable coordinate ϕ.



As before, p_ϕ simply changes ϕ, as shown in red. But now if we trace the action of H,

$$\frac{dr}{dt} = p_r(t)/m, \quad \frac{d\phi}{dt} = \frac{p_\phi}{mr^2},$$

$$\frac{dp_r}{dt} = \frac{p_\phi^2}{mr^3(t)} - kr(t),$$

we get the blue curve which closes on itself after one revolution in ϕ and two trips through the donut hole. Thus the orbit is a closed curve, there is a relation among the frequencies.

Of course the system now only samples the points on the closed curve, so a time average of any function on the trajectory is not the same as the average over the invariant torus.

The 3-D isotropic anharmonic oscillator

Now consider the spherically symmetric oscillator for which the potential energy is not purely harmonic, say $U(r) = \frac{1}{2}kr^2 + cr^4$. Then the Hamiltonian in spherical coordinates is

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + \frac{1}{2}kr^2 + cr^4.$$

This is time independent, so $F_1 = H$ is conserved, the first of our integrals of the motion. Also ϕ is an ignorable coordinate, so $F_2 = p_\phi = L_z$ is the second. But we know that all of $\vec{L}$ is conserved. While L_x is an integral of the motion, it is not in involution with L_1, as $[L_1, L_2] = L_3 \neq 0$, so it will not serve as an additional generator. But $L^2 = \sum_k L_k^2$ is also conserved and has zero Poisson bracket with H and L_z, so we can take it to be the third generator

$$\begin{aligned} F_3 = L^2 &= (\vec{r} \times \vec{p})^2 = r^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2 = r^2\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - r^2p_r^2 \\ &= p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}. \end{aligned}$$

The full phase space is six dimensional, and as p_ϕ is constant we are left, in general, with a five dimensional space with two nonlinear constraints. On the three-dimensional hypersurface, p_ϕ generates motion only in ϕ, the Hamiltonian generates the dynamical trajectory with changes in r, p_r, θ, p_θ and ϕ, and F_3 generates motion in θ, p_θ and ϕ, but not in r or p_r.

Now while L_x is not in involution with the three F_i already chosen, it is a constant of the (dynamical) motion, as $[L_x, H] = 0$. But under the flow generated by $F_2 = L_z$, which generates changes in η_j proportional to $[\eta_j, L_z]$, we have

$$\begin{aligned} \frac{d}{d\lambda} L_x(g^{\lambda L_z}\vec{\eta}) &= \sum_j \frac{\partial L_x(\eta)}{\partial \eta_j}[\eta_j, L_z] = \sum_{jk} \frac{\partial L_x(\eta)}{\partial \eta_j} J_{jk} \frac{\partial L_z}{\eta_k} \\ &= [L_x, L_z] \neq 0. \end{aligned}$$

Thus the constraint on the dynamical motion that L_x is conserved tells us that motion on the invariant torus generated by L_z is inconsistent with the dynamical evolution — that the trajectory lies in a discrete subspace (two dimensional) rather than being dense in the three-dimensional invariant torus. This also shows that there must be one relation among the frequencies.

Of course we could have reached this conclusion much more easily, as we did in section **??**, by choosing the z-axis of the spherical coordinates along whatever direction $\vec{L}$ points, so the motion restricts $\vec{r}$ to the xy plane, and throwing in p_r gives us a two-dimensional torus on which the motion remains.