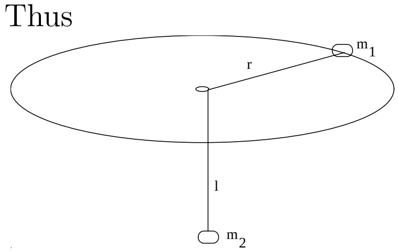
0.1 A particle of mass m_1 lies on a frictionless horizontal table with a tiny hole in it. An inextensible massless string attached to m_1 goes through the hole and is connected to another particle of mass m_2, which moves vertically only. Give a full set of generalized unconstrained coordinates and write the Lagrangian in terms of these. Assume the string remains taut at all times and that the motions in question never have either particle reaching the hole, and there is no friction of the string sliding at the hole.
Are there ignorable coordinates? Reduce the problem to a single second order differential equation.

The length of the string, $r + \ell = \mathcal{L}$ is a constraint, so if we use polar coordinates r and θ for the mass on the table, the remaining coordinate, the height of the hanging mass, is determined. Thus



$$\begin{aligned} T &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{\ell}^2 \\ &= \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2, \\ U &= K_1 - m_2 g\ell = K + m_2 g r, \end{aligned}$$

where K_1 and K are constants (with $K = K_1 - m_2 g\mathcal{L}$). Thus Lagrange's equations are

$$\begin{aligned} (m_1+m_2)\ddot{r} - m_1 r\dot{\theta}^2 + m_2 g &= 0, \\ \frac{d}{dt}m_1 r^2\dot{\theta} = 0. \end{aligned}$$

The second equation tells us that

$$L := m_1 r^2\dot{\theta},$$

which is the angular momentum about the vertical through the hole, is conserved. Then we can rewrite

$$\dot{\theta} = \frac{L}{m_1 r^2}$$

and the first equation becomes

$$(m_1+m_2)\ddot{r} - \frac{L^2}{m_1}r^{-3} + m_2 g = 0.$$

Here we have an effective force $F = L^2/m_1 r^3 - m_2 g$ and an effective mass $M = m_1 + m_2$, and the problem can be described in terms of an effective potential $V(r) = (L^2/2m_1 r^2) + m_2 g r$, and the total energy $E(r, \dot{r}) = \frac{1}{2} M \dot{r}^2 + V(r)$ is a conserved, first integral of the motion.
