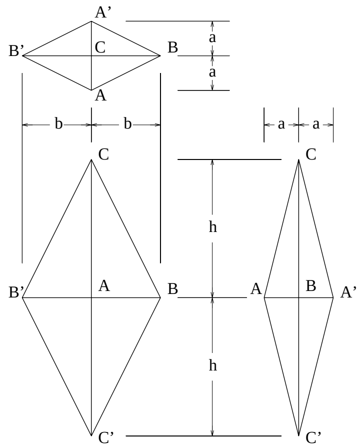
Physics 507 Homework Solutions #6
Due: Thursday, Oct. 14, 2010

6.1 A diamond shaped object is shown in top, front, and side views. It is an octahedron, with 8 triangular flat faces.

It is made of solid aluminum of uniform density, with a total mass M. The dimensions, as shown, satisfy $h > b > a$.
(a) Find the moment of inertia tensor about the center of mass, clearly specifying the coordinate system chosen.
(b) About which lines can a stable spinning motion, with fixed $\vec{\omega}$, take place, assuming no external forces act on the body?



Solution 6.1: Let us take the origin at the center of the body, with the x-axis towards A, the y-axis towards B, and the z-axis towards C. The body is clearly symmetrical under reflection about the yz-plane, under reflection about the xz-plane, and under reflection about the xy-plane, or equivalently under $x \to -x$, under $y \to -y$, and under $z \to -z$. Therefore integrands which are odd under any of these vanish, so

$$\vec{R} = \int d^3r\, \rho(r)\vec{r} = 0, \qquad I_{ij} = \int d^3r\, \rho(r) r_i r_j = 0 \ \text{ for } i \neq j.$$

For integrals symmetric under these reflections, the answer is eight times the integral over the first octant, $x > 0, y > 0, z > 0$, where the body is bounded by the plane passing through $(a, 0, 0)$ and $(0, b, 0)$ and $(0, 0, h)$. As a plane is determined by a linear equation, in this case it is $x/a + y/b + z/h = 1$. The integration region for one octant can therefore be written as

$$\int d^3r\, \rho(\vec{r}) = \rho \int_0^h dz \int_0^{b(1-z/h)} dy \int_0^{a(1-z/h-y/b)} dx.$$

Against this integration measure we need the integrals of 1 (to evaluate the total mass), x^2, y^2, and z^2. For 1 and z^2, the inner two integrals are just the area of a triangle of sides $a(1 - z/h)$ and $b(1 - z/h)$, giving

$$\int_0^{b(1-z/h)} dy \int_0^{a(1-z/h-y/b)} dx\ 1 = \int_0^{b(1-z/h)} dy[a(1 - z/h) - ay/b]$$
$$= \left(a(1 - z/h)y - \frac{1}{2}ay^2/b\right)\Big|_0^{y=b(1-z/h)} = \frac{1}{2}ab(1 - z/h)^2,$$

so

$$\begin{aligned} M = \int d^3r\, \rho(\vec{r}) &= 8 \times \frac{1}{2}ab \int_0^h dz\, (1 - z/h)^2 = \frac{4abh}{3} \\ \int d^3r\, \rho(\vec{r}) z^2 &= 8 \times \frac{1}{2}\rho ab \int_0^h dz\, z^2(1 - z/h)^2 \\ &= 4\rho ab \left(\frac{z^3}{3} - \frac{2z^4}{4h} + \frac{z^5}{5h^2}\right)\Big|_0^h = \frac{2}{15}abh^3 = \frac{M}{10}h^2. \end{aligned}$$

By the distorted symmetry of the problem, we see that the integrals of x^2 and y^2 are of the same form, especially if we reordered the sequence of iterated integrals. Thus $\int d^3r\, \rho(\vec{r})x^2 = Ma^2/10$ and $\int d^3r\, \rho(\vec{r})y^2 = Mb^2/10$, so

$$\begin{aligned} I_{xx} = \int d^3r\, \rho(\vec{r})(r^2 - x^2) &= \int d^3r\, \rho(\vec{r})(y^2 + z^2) = M(b^2 + h^2)/10, \\ I_{yy} &= \int d^3r\, \rho(\vec{r})(x^2 + z^2) = M(a^2 + h^2)/10, \\ I_{zz} &= \int d^3r\, \rho(\vec{r})(x^2 + y^2) = M(a^2 + b^2)/10, \end{aligned}$$

so

$$I_{zz} < I_{yy} < I_{xx}.$$

(b) We saw that torque-free motion with a constant $\vec{\omega}$ requires $\vec{L}$ and $\vec{\omega}$ to be aligned along one of the principle axes, which in this case are just the x-, y-, and z-axes. From the analysis of torque-free rotation with $\vec{L}$ deviating slightly from a principle axis, we saw that stable motion exists about the axes with the largest and smallest moments, so about the x- and z-axes, while spinning about the middle, or y-axis, is unstable.

6.2 We defined the general rotation as $A = R_z(\psi) \cdot R_y(\theta) \cdot R_z(\phi)$. Work out the full expression for $A(\phi, \theta, \psi)$, and verify the last expression in (4.31). [For this and exercise 6.3, you might want to use a computer algebra program such as mathematica or maple, if one is available.]

Solution 6.2: This problem is most simply done by a computer algebra program. Here is what I used, using Mathematica

```
OpenWrite["logfile"]                Open a log file so that if we don't
PrependTo[$Echo,"logfile"]          get it perfect the first time,
                                    we can read in the log file.
<<Algebra`Trigonometry`             Needed for TrigReduce below
Rz[phi_] =
{{Cos[phi],Sin[phi],0},
```

`{-Sin[phi],Cos[phi],0},`	defines the matrix
`{0,0,1}}`	$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$
`Ry[theta_]=`	
`{{Cos[theta],0,-Sin[theta]},`	
`{0,1,0},`	Defines the matrix
`{Sin[theta],0,Cos[theta]}}`	$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$
`A = Simplify[Rz[psi[t]]`	$A = R_z(\psi(t)) \cdot R_y(\theta(t)) \cdot R_z(\phi(t))$
`. Ry[theta[t]]`	The Simplify is needed to
`. Rz[phi[t]]]`	make Mathematica do the work.
`Ainv = Transpose[A]`	Remember $A^{-1} = A^T$, which is easier
`Ainvdot = D[Ainv,t]`	$dA^{-1}(t)/dt$
`OmA = - Simplify[`	
`A . Ainvdot]`	This is Ω_{ij}
`Check = Simplify[`	should be, and is, 0
`OmA + Transpose[OmA]]`	
`Array[omega,3]`	I think this is needed to declare the vector, maybe not
`omega[1]=Expand`	
`Reduce[OmA[[2,3]]]]`	$\omega_1 = \Omega_{23}$
`omega[2]=Expand`	
`Reduce[OmA[[3,1]]]]`	$\omega_2 = \Omega_{31}$
`omega[3]=Expand`	
`Reduce[OmA[[1,2]]]]`	$\omega_3 = \Omega_{12}$
`atex=TeXForm[A]`	Putting these in TeX form didn't
`omtex=TeXForm[omega]`	help much, and these had to be editted by hand to look good.
`Save["omega.out",`	Save the results in the file
`omtex,atex]`	"`omega.out`"

Here are the results, prettied up: $A(\phi,\theta,\psi) =$

$$\begin{pmatrix} -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & \cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi & -\sin\theta\cos\psi \\ -\sin\phi\cos\psi - \cos\theta\cos\phi\sin\psi & \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\theta\sin\psi \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}.$$

$$\begin{aligned} \omega_1 &= \Omega_{23} = \dot\theta\sin\psi - \dot\phi\sin\theta\cos\psi, \\ \omega_2 &= \Omega_{31} = \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi, \\ \omega_3 &= \Omega_{12} = \dot\psi + \dot\phi\cos\theta. \end{aligned}$$

6.3 Find the expression for $\vec\omega$ in terms of $\phi,\theta,\psi,\dot\phi,\dot\theta,\dot\psi$. [This can be done simply with computer algebra programs. If you want to do this by hand, you might find it easier to use the product form $A = R_3R_2R_1$, and the rather simpler expressions for $R\dot R^T$. You will still need to bring the result (for $R_1\dot R_1^T$, for example) through the other rotations, which is somewhat messy.]

Solution 6.3: (See solution to (6.2))
