Physics 507 Homework Solutions #5
Due: Thursday, Oct. 7, 2010

5.1 Consider a particle constrained to move on the surface described in cylindrical coordinates by $z = \alpha r^3$, subject to a constant gravitational force $\vec{F} = -mg\hat{e}_z$. Find the Lagrangian, two conserved quantities, and reduce the problem to a one dimensional problem. What is the condition for circular motion at constant r?

Solution 5.1: The potential energy is $U = mgz = mg\alpha r^3$, while the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[1 + (3\alpha r^2)^2\right]\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2,$$

so

$$L = \frac{1}{2}m\left(1 + 9\alpha^2 r^4\right)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mg\alpha r^3.$$

The coordinate θ is ignorable, so

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \text{ is conserved.}$$

The Hamiltonian

$$\begin{aligned} H &= P_\theta\dot{\theta} + P_r\dot{r} - L = \frac{P_\theta^2}{mr^2} + m\left(1 + 9\alpha^2 r^4\right)\dot{r}^2 - L \\ &= \frac{P_\theta^2}{2mr^2} + \frac{1}{2}m\left(1 + 9\alpha^2 r^4\right)\dot{r}^2 + mg\alpha r^3 \end{aligned}$$

is also conserved, because $dH/dt = -\partial L/\partial t = 0$. Considering P_θ as a fixed constant, this gives us a one dimensional problem in r. The equation of motion follows either from Lagrange's equation in the r coordinate or from $dH/dt = 0$,

$$m\left(1 + 9\alpha^2 r^4\right)\ddot{r} + 18m\alpha^2 r^3\dot{r}^2 - \frac{P_\theta^2}{mr^3} + 3mg\alpha r^2 = 0.$$

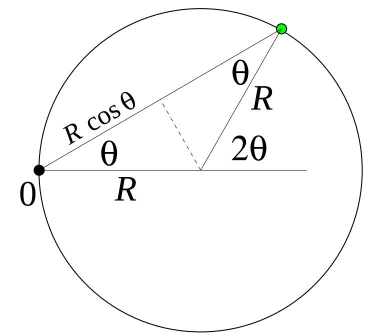
For circular motion at a constant r, $\dot{r} = \ddot{r} = 0$, so $P_\theta^2/mr^3 = 3mg\alpha r^2$, or $P_\theta = m\sqrt{3g\alpha r^5}$.

5.2 Suppose a particle of mass m moves under the influence of a power-law central force, $\vec{F} = -cr^p\hat{e}_r$, and is observed to have an orbit which is a circle of radius R *passing through* the point of attraction.

Find what values the power p could be, what is the angular momentum about the *center of force*, and what is the energy relative to $U(\infty)$.

How do $\dot{\theta}$, $\dot{y}$, and $\dot{x}$ behave as the particle approaches the origin, as a function of r as $r \to 0$?. Is this consistent with x taking its minimum value at that point?

Solution 5.2: Take the center of the circular orbit to be at $\rho = R, \theta = 0$ in polar coordinates, so an arbitrary point on the circular orbit is at $(\rho{=}2R\cos\theta, \theta)$ in polar coordinates and $(x{=}R{+}R\cos 2\theta, y{=}R\sin 2\theta)$ in cartesian coordinates. The angular momentum about the origin is $L_z = m\rho^2\dot{\theta} = 4mR^2\cos^2\theta\,\dot{\theta}$, where the first expression is as usual, but could also have been found from $L_z = m(x\dot{y} - y\dot{x})$ using



$$\dot{x} = -2R\sin(2\theta)\dot{\theta}, \qquad \dot{y} = -2R\cos(2\theta)\dot{\theta}.$$

Of course L_z is conserved, a constant, as we have a central force here. The constant energy is

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + U(\rho) = 2mR^2\dot{\theta}^2 + U(\rho).$$

For the central force $\vec{F} = -cr^p\hat{e}_r$, $U(|\vec{r}|) = \frac{c}{p+1}|\vec{r}|^{p+1}$, so

$$E = 2mR^2\dot{\theta}^2 + U(2R\cos\theta) = 2mR^2\left(\frac{L_z}{4mR^2\cos^2\theta}\right)^2 + \frac{c}{p+1}\left(2R\cos\theta\right)^{p+1},$$

which is only possible for all $|\theta| \le \pi/2$ only if $p = -5$, $E = 0$. And then

$$\frac{2mR^2L_z^2}{(4mR^2\cos^2\theta)^2} = \frac{c}{4}\frac{1}{(2R\cos\theta)^4} \Longrightarrow L_z = \sqrt{\frac{cm}{8}}\,\frac{1}{R}.$$

Of course in order to have a finite angular momentum about the origin as the particle passes through the origin, the velocity must blow up dramatically. In terms of the polar distance ρ, $\dot{\theta}$ must blow up quadratically (*i.e.* $\dot{\theta} \sim \rho^{-2}$), which must also be true for $\dot{y}$. Most surprising is the behavior of x as the particle passes through the origin. As x takes its minimum value there, we might expect that $x = \dot{x} = 0$ at that moment, but it is not so. In fact, $\dot{x} = -2R\sin(2\theta)\dot{\theta}$ blows up, because $\rho \approx y = R\sin 2\theta$ vanishes linearly, while $\dot{\theta}$ blows up quadratically, so dx/dt changes sign my going through ∞, not zero!

Hamilton's Principle tells us that the motion of a particle is determined by the action functional being stationary under small variations of the path Γ in extended configuration space $(t, \vec{x})$. The unsymmetrical treatment of t and $\vec{x}(t)$ is not suitable for relativity, but we may still associate an action with each path, which we can parameterize with λ, so Γ is the trajectory $\lambda \to (t(\lambda), \vec{x}(\lambda))$.

In the general relativistic treatment of a particle's motion in a gravitational field, the action is given by $mc^2\Delta\tau$, where $\Delta\tau$ is the elapsed proper time, $\Delta\tau = \int d\tau$. But distances and time intervals are measured with a spatial varying metric $g_{\mu\nu}$, with μ and ν ranging from 0 to 3, with the zeroth component referring to time. The four components of extended configuration space are written x^μ, with a superscript rather than a subscript, and $x^0 = ct$. The other three x^μ can be generalized coordinates, as long as $g_{\mu\nu}$ is appropriate. In the next problem they are similar to spherical coordinates. The gravitational field is described by the space-time dependence of the metric $g_{\mu\nu}(x^\rho)$. In this language, an infinitesimal element of the path of a particle corresponds to a proper time $d\tau = (1/c)\sqrt{\sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu}$, so

$$S = mc^2\Delta\tau = mc\int d\lambda \sqrt{\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}.$$

This is in preparation for the next problem *which is worth 20 points, twice normal.*

5.3 In problem 2.12 we learned that the general-relativistic motion of a particle in a gravitational field is given by Hamilton's variational principle on the path $x^\mu(\lambda)$ with the action

$$S = \int d\lambda \mathcal{L} \quad \text{with} \quad \mathcal{L} = mc\sqrt{\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}},$$

where we may freely choose the path parameter λ to be the proper time (after doing the variation), so that the $\sqrt{\ }$ is c, the speed of light.

The gravitational field of a static point mass M is given by the Schwartzschild metric

$$g_{00} = 1-\frac{2GM}{rc^2}, \quad g_{rr} = -1\Big/\left(1-\frac{2GM}{rc^2}\right), \quad g_{\theta\theta} = -r^2, \quad g_{\phi\phi} = -r^2\sin^2\theta,$$

where all other components of $g_{\mu\nu}$ are zero. Treating the four $x^\mu(\lambda)$ as the coordinates, with λ playing the role of time, find the four conjugate momenta p_μ, show that p_0 and $p_\phi = L$ are constants, and use the freedom to choose

$$\lambda = \tau = \frac{1}{c}\int\sqrt{\sum_{\mu\nu} g_{\mu\nu}(x^\rho)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

to show $m^2c^2 = \sum_{\mu\nu} g^{\mu\nu} p_\mu p_\nu$, where $g^{\mu\nu}$ is the inverse matrix to $g_{\alpha\beta}$. Use this to show that

$$\frac{dr}{d\tau} = \sqrt{\kappa - \left(-\frac{2GM}{r} + \frac{L^2}{m^2r^2} - \frac{2GML^2}{m^2r^3c^2}\right)},$$

where κ is a constant. For an almost circular orbit at the minimum $r = a$ of the effective potential this implies, show that the precession of the perihelion is $6\pi GM/ac^2$.

Find the rate of precession for Mercury, with $G = 6.67\times 10^{-11}\,\mathrm{Nm^2/kg^2}$, $M = 1.99\times 10^{30}\,\mathrm{kg}$ and $a = 5.79\times 10^{10}\,\mathrm{m}$, per revolution, and also per century, using the period of the orbit as 0.241 years.

Solution 5.3: Defining the (four) conjugate momenta as usual by

$$p_\mu := \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = mc\frac{\sum_\nu g_{\mu\nu}(x)\dot{x}^\nu}{\sqrt{\sum_{\alpha\beta} g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta}} \to m\sum_\nu g_{\mu\nu}(x)\dot{x}^\nu$$

where dot means $d/d\lambda$, and the arrow means after setting λ to be the proper time. Defining $g^{\mu\nu}$ to be the inverse matrix to $g_{\alpha\beta}$, we then have $\dot{x}^\mu = \sum_\nu g^{\mu\nu}p_\nu/m$. Lagrange's equations gives us

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{d}{d\tau}p_\mu = \frac{\partial\mathcal{L}}{\partial x^\mu} = mc\sum_{\alpha\beta}\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{\dot{x}^\alpha\dot{x}^\beta}{\sqrt{\sum g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \to \frac{m}{2}\sum_{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\dot{x}^\alpha\dot{x}^\beta.$$

For the Schwartzschild metric, no $g_{\alpha\beta}$ depends on $x^0 = ct$ or on ϕ, so these are ignorable coordinates and the momenta p_0 and p_ϕ are constants, which we will call $-E$ and $\pm L$ respectively (choosing $L \geq 0$). In the equatorial plane, $\theta = \pi/2$, all the $\partial g_{\alpha\beta}/\partial\theta$ are zero, as the only one not trivially so is $\partial g_{\phi\phi}/\partial\theta = -2r^2\sin\theta\cos\theta \to 0$. Thus $dp_\theta/d\tau = 0$, and if a particle starts off with $\theta = \pi/2$, $p_\theta = 0$ this will remain true, the motion will be restricted to the equatorial plane, just as in non-relativistic motion.

Now the condition that τ is the proper time is

$$\begin{aligned} m^2c^2 &= m^2\sum_{\alpha\beta} g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta = \sum_{\alpha\beta} g^{\alpha\beta}p_\alpha p_\beta \\ &= \frac{E^2}{1-\frac{2GM}{rc^2}} - \left(1-\frac{2GM}{rc^2}\right)p_r^2 - \frac{L^2}{r^2} = \frac{E^2 - m^2\dot{r}^2}{1-\frac{2GM}{rc^2}} - \frac{L^2}{r^2} \end{aligned}$$

Thus

$$\begin{aligned}
\frac{dr}{d\tau} &= \sqrt{\frac{E^2}{m^2} - \left(c^2 + \frac{L^2}{m^2 r^2}\right)\left(1 - \frac{2GM}{rc^2}\right)} \\
&= \sqrt{\frac{E^2}{m^2} - c^2 - \left(-\frac{2GM}{r} + \frac{L^2}{m^2 r^2} - \frac{2GML^2}{m^2 r^3 c^2}\right)}.
\end{aligned}$$

Compare to the non-relativistic expression

$$\text{NR:} \qquad \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E_{\text{NR}} + \frac{GMm}{r} - \frac{L^2}{2mr^2}\right)}.$$

We see that aside from the constant no longer being called the energy and the time being replaced by proper time, the only effect of general relativity is to add the r^{-3} term to the effective potential (per unit mass)

$$U_{\text{eff}}(r) = -\frac{GM}{r} + \frac{\tilde{L}^2}{2r^2} - \frac{GM\tilde{L}^2}{r^3 c^2},$$

where $\tilde{L} = L/m$ is the angular momentum per unit mass, and we still have $|\dot{\phi}| = |L/mg_{\phi\phi}| = \tilde{L}/r^2$, so the question of precession of the perihelion is as before, but with an effective potential with a small negative $1/r^3$ term.

To calculate the precession of Mercury, we can assume this last term is small, and that the orbit is close to circular, near the minimum of $U_{\text{eff}}(r)$, so we evaluate

$$\frac{dU_{\text{eff}}(r)}{dr} = \frac{GM}{r^2} - \frac{\tilde{L}^2}{r^3} + 3\frac{GM\tilde{L}^2}{r^4 c^2},$$

so the minimum of $U_{\text{eff}}(r)$ is at $r = a$ where

$$GMa^2 - \tilde{L}^2 a + 3\frac{GM\tilde{L}^2}{c^2} = 0,$$

or

$$\begin{aligned}
a &= \frac{\tilde{L}^2 \pm \sqrt{\tilde{L}^4 - 12G^2M^2\tilde{L}^2/c^2}}{2GM} \to \frac{\tilde{L}^2}{2GM}\left(1 \pm (1 - 6G^2M^2/c^2\tilde{L}^2)\right) \\
&\to \frac{\tilde{L}^2}{GM}\left(1 - 3\frac{G^2M^2}{c^2\tilde{L}^2}\right) = \frac{\tilde{L}^2}{GM} - 3\frac{GM}{c^2}, \quad \text{so} \quad \frac{\tilde{L}^2}{GMa} = 1 + 3\frac{GM}{ac^2}.
\end{aligned}$$

The first $\to$ makes use of the smallness of $GM/c\tilde{L}$, and for the second we choose the plus sign. The other sign shows that the potential turns strongly attractive for very small r, which corresponds to falling into the black hole, but Mercury is in no danger of getting past that potential barrier.

The second derivative of $U_{\text{eff}}(r)$ gives the effective spring constant,

$$\begin{aligned}
k &= \left.\frac{d^2U_{\text{eff}}}{dr^2}\right|_a = -2\frac{GM}{a^3} + 3\frac{\tilde{L}^2}{a^4} - 12\frac{GM\tilde{L}^2}{a^5c^2} \\
&\to \frac{GM}{a^3}\left(-2 + 3\frac{\tilde{L}^2}{GMa} - 12\frac{\tilde{L}^2}{a^2c^2}\right) \\
&= \frac{GM}{a^3}\left(-2 + 3\left(1 + 3\frac{GM}{ac^2}\right) - 12\left(1 + 3\frac{GM}{ac^2}\right)\frac{GM}{ac^2}\right) \\
&= \frac{GM}{a^3}\left(1 - 3\frac{GM}{ac^2}\right)
\end{aligned}$$

so the period of oscillation is

$$T_{\text{osc}} = 2\pi\sqrt{2/k} \approx 2\pi\sqrt{\frac{a^3}{GM}}\left(1 + \frac{3}{2}\frac{GM}{ac^2}\right)$$

while the period of revolution is $T_{\text{rev}} = 2\pi/\dot{\phi} = 2\pi a^2/\tilde{L}$, but as $\tilde{L}^2 = GMa\,(1 + 3GM/ac^2)$, this is

$$T_{\text{rev}} = 2\pi\sqrt{\frac{a^3}{GM}}\left(1 - \frac{3}{2}\frac{GM}{ac^2}\right)$$

The precession of the perihelion is

$$2\pi\left(\frac{T_{\text{osc}}}{T_{\text{rev}}} - 1\right) = 2\pi \times \frac{3GM}{c^2 a}.$$

With $G = 6.67 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$, $M = 1.99 \times 10^{30}\,\text{kg}$ and $a = 5.79 \times 10^{10}\,\text{m}$, the precession is $4.81 \times 10^{-7}\,\text{rad} = 0.0992''$ per revolution. The period of Mercury's orbit is 0.241 years, so in a century this is 415 revolutions for a total precession of 41.1 seconds of arc per century.