Physics 507 Homework Solutions #10
Due: Thursday, Nov. 18, 2010

10.1 [20 points] We have considered k-forms in 3-D Euclidean space and their relation to vectors expressed in cartesian basis vectors. We have seen that k-forms are invariant under change of coordinatization of $\mathcal{M}$, so we can use them to examine the forms of the gradient, curl, divergence and laplacian in general coordinates in three dimensional space. We will restrict our treatment to *orthogonal curvilinear coordinates* (q_1, q_2, q_3), for which we have, at each point $\mathbf{p} \in \mathcal{M}$, a set of **orthonormal** basis vectors $\hat{e}_i$ directed along the corresponding coordinate, so that $dq_i(\hat{e}_j) = 0$ for $i \neq j$. We assume they are right handed, so $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ and $\hat{e}_i \times \hat{e}_j = \sum_k \epsilon_{ijk}\hat{e}_k$. The dq_i are not normalized measures of distance, so we define $h_i(\mathbf{p})$ so that $dq_i(\hat{e}_j) = h_i^{-1}\delta_{ij}$ (no sum).

(a) For a function $f(q_1, q_2, q_3)$ and a vector $\vec{v} = \sum v_i\hat{e}_i$, we know that $df(\vec{v}) = \vec{v} \cdot \vec{\nabla} f$. Use this to find the expression for $\vec{\nabla} f$ in the basis $\hat{e}_i$.
(b) Use this to get the general relation of a 1-form $\sum \omega_i dq_i$ to its associated vector $\vec{v} = \sum v_i\hat{e}_i$.
(c) If a 1-form $\omega^{(a)}$ is associated with $\vec{v}^{(a)}$ and 1-form $\omega^{(b)}$ is associated with $\vec{v}^{(b)}$, we know the 2-form $\omega^{(a)} \wedge \omega^{(b)}$ is associated with $\vec{v}^{(a)} \times \vec{v}^{(b)}$. Use this to find the general association of a 2-form with a vector.
(d) We know that if a 1-form ω is associated with a vector $\vec{v}$, then $d\omega$ is associated with $\vec{\nabla} \times \vec{v}$. Use this to find the expression for $\vec{\nabla} \times \vec{v}$ in orthogonal curvilinear coordinates.
(e) If the 1-form ω is associated with $\vec{v}$ and the 2-form Ω is associated with $\vec{F}$, we know that $\omega \wedge \Omega$ is associated with the scalar $\vec{v} \cdot \vec{F}$. Use this to find the general association of a 3-form with a scalar.
(f) If the 2-form Ω is associated with $\vec{v}$, we know that $d\Omega$ is associated with the divergence of $\vec{v}$. Use this to find the expression for $\vec{\nabla} \cdot \vec{v}$ in orthogonal curvilinear coordinates.
(g) Use (a) and (f) to find the expression for the laplacian of a scalar, $\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f$, in orthogonal curvilinear coordinates.

Solution 10.1: (a) $df(\vec{v}) = \sum v_j \dfrac{\partial f}{\partial q_i} dq_i(\hat{e}_j) = \sum h_i^{-1} v_i \dfrac{\partial f}{\partial q_i}$, so

$$\vec{\nabla} f = \sum h_i^{-1} \frac{\partial f}{\partial q_i} \hat{e}_i.$$

(b) As $df = \sum \dfrac{\partial f}{\partial q_i} dq_i$ is associated with $\sum h_i^{-1} \dfrac{\partial f}{\partial q_i} \hat{e}_i$, we may more generally associate a 1-form $\sum \omega_i dq_i$ with $\vec{v} = h_i^{-1}\omega_i \hat{e}_i$. Also we can note $\sum_i \omega_i dq_i(\hat{e}_j) = \omega_j/h_j$, so $dq_i(\hat{e}_j) = \delta_{ij}/h_j$.

(c) $\omega^{(a)} \wedge \omega^{(b)} = \sum_{ij} \omega_i^{(a)} \omega_j^{(b)} dq_i \wedge dq_j = \sum_{ij} B_{ij} dq_i \otimes dq_j$ with $B_{ij} = \omega_i^{(a)}\omega_j^{(b)} - \omega_i^{(b)}\omega_j^{(a)}$. The vector[1] $\vec{v}^{(a)} \times \vec{v}^{(b)} = \sum_{ijk} \epsilon_{ijk} v_i^{(a)} v_j^{(b)} \hat{e}_k = \sum_{ijk} \epsilon_{ijk} (h_i h_j)^{-1} \omega_i^{(a)} \omega_j^{(b)} \hat{e}_k = \vec{C} = \sum_k C_k \hat{e}_k$, where $C_k = \sum_{ij} \epsilon_{ijk} (h_i h_j)^{-1} \omega_i^{(a)} \omega_j^{(b)}$. More generally, a two form $\mathbf{B} = \sum B_{ij} dq_i \otimes dq_j$ is associated with $C_k = \frac{1}{2}\sum_{ij} \epsilon_{ijk}(h_i h_j)^{-1} B_{ij}$, or $B_{ij} = \sum_{ij} \epsilon_{ijk} h_i h_j C_k$.

(d) With $\omega = \sum \omega_i dq_i$ associated with $\vec{v} = \sum v_i \hat{e}_i$, which requires $\omega_i = h_i v_i$, we have $d\omega = \displaystyle\sum_{ij} \frac{\partial h_i v_i}{\partial q_j} dq_j \wedge dq_i = \sum_{ij} \left(\frac{\partial h_j v_j}{\partial q_i} - \frac{\partial h_i v_i}{\partial q_j} \right) dq_i \otimes dq_j$, which is associated with $\vec{C} = \vec{\nabla} \times \vec{v} = \sum C_k \hat{e}_k$ with

$$C_k = \frac{1}{2} \sum_{ij} \epsilon_{ijk} (h_i h_j)^{-1} \left(\frac{\partial h_j v_j}{\partial q_i} - \frac{\partial h_i v_i}{\partial q_j} \right) = \sum_{ij} \epsilon_{ijk} (h_i h_j)^{-1} \frac{\partial h_j v_j}{\partial q_i}.$$

(e) If $\omega = \sum_\ell h_\ell v_\ell \, dq_\ell$, and

$$\Omega = \sum_{ijk} \epsilon_{ijk} h_i h_j F_k dq_i \otimes dq_j = \frac{1}{2} \sum_{ijk} \epsilon_{ijk} h_i h_j F_k dq_i \wedge dq_j,$$

we have

$$\begin{aligned}
\omega \wedge \Omega &= \frac{1}{2} \sum_{ijk\ell} h_\ell v_\ell \epsilon_{ijk} h_i h_j F_k dq_\ell \wedge dq_i \wedge dq_j \\
&= \frac{1}{2} \sum_{ijk\ell} \epsilon_{ijk} \epsilon_{\ell ij} h_\ell v_\ell h_i h_j F_k \, dq_1 \wedge dq_2 \wedge dq_3 \\
&= \left(\prod_{j=1}^{3} h_j \right) \sum_k v_k F_k \, dq_1 \wedge dq_2 \wedge dq_3.
\end{aligned}$$

This corresponds to the scalar $\vec{v} \cdot \vec{F} = \sum_k v_k F_k$, so more generally a 3-form $\phi \, dq_1 \wedge dq_2 \wedge dq_3$ corresponds to the scalar field $\phi / \prod_1^3 h_i$.

(f) With $\Omega = \frac{1}{2} \sum_{ijk} \epsilon_{ijk} h_i h_j v_k dq_i \wedge dq_j$,

$$\begin{aligned}
d\Omega &= \frac{1}{2} \sum_{ijk\ell} \epsilon_{ijk} \frac{\partial h_i h_j v_k}{\partial q_\ell} \, dq_\ell \wedge dq_i \wedge dq_j \\
&= \frac{1}{2} \sum_{ijk\ell} \epsilon_{ijk} \epsilon_{\ell ij} \frac{\partial h_i h_j v_k}{\partial q_\ell} \, dq_1 \wedge dq_2 \wedge dq_3 \\
&= \sum_\ell \frac{\partial}{\partial q_\ell} \left(\frac{h_1 h_2 h_3 v_\ell}{h_\ell} \right) dq_1 \wedge dq_2 \wedge dq_3
\end{aligned}$$

[1] In differential geometry, or in the discussion of forms without the restriction to orthonormal basis vectors, one introduces the Levi-Civita symbol ε_{ijk} (with D subscripts in D dimensional space), which is proportional to, but not equal to, the flat-space ϵ_{ijk} for which $\epsilon_{123} = 1$. The relation is $\varepsilon_{ijk} = \sqrt{|\det g_{..}|}\epsilon_{ijk}$, where $g_{..}$ is the metric tensor. The volume element is then given by the D-form $\sum \varepsilon_{\mu_1,\ldots\mu_D} dq^{\mu_1} \wedge \cdots \wedge dq^{\mu_D}$. But here we will only use ϵ_{ijk}, not ε_{ijk}. Note the notational distinction I have made is not standard.

which is associated with the scalar $\displaystyle\frac{1}{h_1h_2h_3}\sum_\ell \frac{\partial}{\partial q_\ell}\left(\frac{h_1h_2h_3v_\ell}{h_\ell}\right)$.

$$\text{Thus}\qquad \vec\nabla\cdot\vec v = \frac{1}{h_1h_2h_3}\sum_\ell \frac{\partial}{\partial q_\ell}\left(\frac{h_1h_2h_3v_\ell}{h_\ell}\right).$$

(g) As $\vec\nabla f = \sum h_i^{-1}\dfrac{\partial f}{\partial q_i}\hat e_i$ is the vector with coefficients $v_\ell = h_\ell^{-1}\dfrac{\partial f}{\partial q_\ell}$,

$$\vec\nabla\cdot\vec\nabla f = \frac{1}{h_1h_2h_3}\sum_\ell \frac{\partial}{\partial q_\ell}\left(\frac{h_1h_2h_3}{h_\ell^2}\frac{\partial f}{\partial q_\ell}\right)$$

is the laplacian of f.

10.2 Consider the unusual Hamiltonian for a one-dimensional problem

$$H = \omega(x^2+1)p,$$

where ω is a constant.

(a) Find the equations of motion, and solve for $x(t)$.

(b) Consider the transformation to new phase-space variables $P = \alpha p^{\frac12}$, $Q = \beta x p^{\frac12}$. Find the conditions necessary for this to be a canonical transformation, and find a generating function $F(x,Q)$ for this transformation.

(c) What is the Hamiltonian in the new coordinates?

Solution 10.2:

(a) $\dot x = \dfrac{\partial H}{\partial p} = \omega(x^2+1)$, $\dot p = -\dfrac{\partial H}{\partial x} = -2\omega x p$. From the first equation,

$$\int\frac{dx}{x^2+1} = \int \omega dt,\qquad \text{or}\qquad \tan^{-1}x = \omega t+\delta,\qquad \Rightarrow\qquad x = \tan(\omega t+\delta).$$

(b) The new variables $Q = \beta x p^{\frac12}$ and $P = \alpha p^{\frac12}$ are canonical if

$$[Q,P] = 1 = \beta p^{\frac12}\frac{\alpha}{2p^{\frac12}} = \frac{\alpha\beta}{2},$$

so all that is needed is $\beta = 2/\alpha$. For a generating function of type 1, we need to solve $p(x,Q) = Q^2\beta^{-2}x^{-2} = \alpha^2Q^2/4x^2$, but

$$p = \left.\frac{\partial F}{\partial x}\right|_Q \Rightarrow F(x,Q) = -\frac{\alpha^2Q^2}{4x} + f(Q).$$

Then $P = -\partial F/\partial Q = \frac{\alpha^2 Q}{2x} - f'(Q) = \alpha p^{\frac12} - f'(Q) = \alpha p^{\frac12}$, which implies $f'(Q)=0$, and we can drop the unknown constant f.

(c) As the transformation is not time-dependent, the Hamiltonian is obtained simply by substituting $p = P^2/\alpha^2$ and $x = Q/(\beta p^{\frac12}) = \alpha Q/(\beta P) = \frac12\alpha^2 Q/P$. Thus we have

$$H = \omega\left(\frac{\alpha^2 4Q^2}{4P^2}+1\right)\frac{P^2}{\alpha^2} = \omega\left(\frac{\alpha^2Q^2}{4}+\frac{P^2}{\alpha^2}\right).$$

If we choose $\omega = \sqrt{k/m}$ and $\alpha = (4km)^{1/4}$, this becomes

$$H = \frac{k}{2}Q^2 + \frac{1}{2m}P^2,$$

our standard harmonic oscillator. Note if $Q = A\sin(\omega t+\delta)$, $P = Am\dot Q = \frac12\alpha^2 A\cos(\omega t+\delta)$, and $x = \frac12\alpha^2 Q/P = \tan(\omega t+\delta)$ in agreement with our previous solution.
