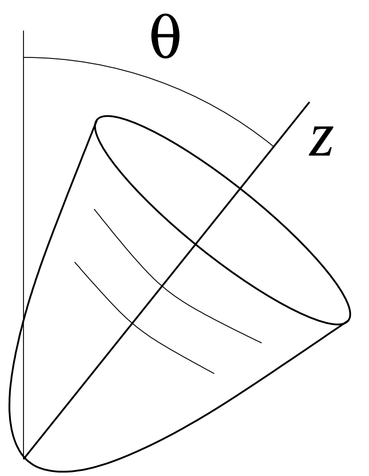
Physics 507 Homework Solution #8

Due: Nov. 4, 2010

1. First, redo the last problem from the midterm:

8.1 Consider a paraboloid defined in polar coordinates (r, ϕ, z) by $z = \alpha r^2$, where α is a positive constant. The paraboloid is fixed in space so that its symmetry axis makes an angle of $\theta < \pi/2$ with respect to the upward vertical direction. This is in the usual gravitational field with $\vec{F} = mg$ downward.

A point particle is constrained to move without friction on the paraboloid. Being careful with the meaning of your variables,



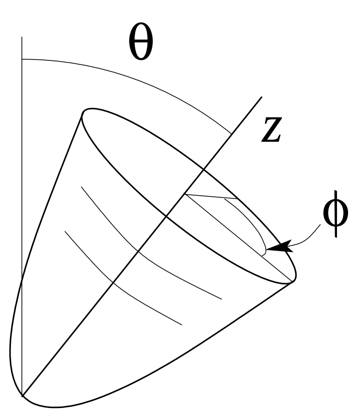
(a) Give the Lagrangian in terms of a set of unconstrained variables.

(b) Are there any conserved quantities? What are they?

(c) There is a stable fixed point. What are its coordinates?

(d) What are the normal modes of small oscillations about the fixed point, and what are the frequencies of each of these modes?

Solution 8.1: Let ϕ be measured with respect to the radius vector most nearly downward, so the height is $h = z\cos\theta - r\sin\theta\cos\phi$, and the potential energy, due entirely to gravity, is $U = mgh$, so



(a)
$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - mg\left(z\cos\theta - r\sin\theta\cos\phi\right).$$

Using r and ϕ as our unconstrained variables, $z = \alpha r^2$ and $\dot{z} = 2\alpha r\dot{r}$, so

$$L = \frac{1}{2}m\left(\dot{r}^2(1+4\alpha^2r^2) + r^2\dot{\phi}^2\right) - mg\left(\alpha r^2\cos\theta - r\sin\theta\cos\phi\right).$$

(b) The Lagrangian has no explicit time dependence, so the Hamiltonian is conserved, and is equal to the energy,

$$E = \frac{1}{2}m\left(\dot{r}^2(1+4\alpha^2r^2) + r^2\dot{\phi}^2\right) + U(r),$$

where $U(r) = mg\left(\alpha r^2\cos\theta - r\sin\theta\cos\phi\right)$.

(c) The stable fixed point is at the minimum of the potential. $\partial U/\partial\phi = 0$ tells us $\phi = 0$ or π, with $\phi = 0$ the *stable* fixed point. Minimizing with respect to r (with $\phi = 0$) gives $2\alpha r\cos\theta = \sin\theta$, so the minimum is at $r_0 = \dfrac{\tan\theta}{2\alpha}$.

(d) Writing $r = r_0 + \eta$, we have to second order in η and $\dot{\eta}$,

$$T = \frac{m}{2}\left[1 + 4\alpha^2\left(\frac{\tan\theta}{2\alpha}\right)^2\dot{\eta}^2 + \left(\frac{\tan\theta}{2\alpha}\right)^2\dot{\phi}^2\right] = \frac{m}{2}\left(\sec^2\theta\,\dot{\eta}^2 + \frac{\tan^2\theta}{4\alpha^2}\dot{\phi}^2\right),$$

and

$$U = mg\left(\alpha\cos\theta\eta^2 + \frac{\tan\theta}{2\alpha}\sin\theta\,\frac{\phi^2}{2}\right) + U_0,$$

where we are free to drop the irrelevant potential U_0 of the fixed point. Note that the coupling of η and ϕ is higher order $\mathcal{O}(\eta\phi^2)$ and vanishes in the small deviation approximation.

Thus the normal modes are oscillation with ϕ fixed at zero, with frequency

$$f_r = \frac{1}{2\pi}\omega_r = \sqrt{\frac{2g\alpha\cos\theta}{sec^2\theta}} = \sqrt{2g\alpha}\cos^{3/2}\theta,$$

and oscillations at constant z with $f_\phi = \omega_\phi/2\pi = \sqrt{2\alpha g\cos\theta}$.

8.2 In considering the limit of a loaded string we found that in the limit $a \to 0, n \to \infty$ with ℓ fixed, the modes with fixed integer p became a smooth excitation $y(x,t)$ with finite wavenumber k and frequency $\omega = ck$.

Now consider the limit with $q := n + 1 - p$ fixed as $n \to \infty$. Calculate the expression for y_j in that limit. This will not have a smooth limit, but there is nonetheless a sense in which it can be described by a finite wavelength. Explain what this is, and give the expression for y_j in terms of this wavelength.

Solution 8.2: The expression for the motion of the j'th mass is

$$y_j(t) = A\sin(k_p ja)\cos(\omega t),$$

with

$$k_p = \frac{p\pi}{(n+1)a} = \frac{\pi}{a}\frac{n+1-q}{n+1} = \frac{\pi}{a} - \frac{q\pi}{\ell}.$$

Thus

$$y_j = A\sin\left(\pi j - \frac{q\pi ja}{\ell}\right) = (-1)^{j+1}A\sin\left(\frac{q\pi x}{\ell}\right)\cos(\omega t).$$

The displacements change sign from mass to nearest neighbor, but the amplitude of this oscillation has a smooth envelope $\sin q\pi x/\ell$ with a finite wavelength $2\ell/q$.

The frequency of this oscillation does go to infinity as $a \to 0$, which more appropriately may be considered

$$\omega = 2\sqrt{\frac{\tau}{\rho}}\sin\left(\frac{n+1-q}{n+1}\frac{\pi}{2}\right)\frac{1}{a} \to 2\sqrt{\frac{\tau}{\rho}}\sin\left(\frac{\pi}{2}\right)\frac{1}{a} = \frac{2}{a}\sqrt{\frac{\tau}{\rho}}.$$

8.3 Consider the Navier equation ignoring the volume force, and show that
a) a uniform elastic material can support longitudinal waves. At what speed do they travel?
b) an uniform elastic material can support transverse waves. At what speed do they travel?
c) Granite has a density of 2700 $\mathrm{kg/m^3}$, a bulk modulus of $4\times10^{10}\mathrm{N/m^2}$ and a shear modulus of $2.5\times10^{10}\mathrm{N/m^2}$. If a short spike of transverse oscillations arrives 25 seconds after a similar burst of longitudinal oscillations, how far away was the explosion that caused these waves?

Solution 8.3: The Navier equation assumes an isotropic medium, so we might as well assume we are looking for a plane wave in the z direction,

$$\vec\eta(\vec r) = \vec A e^{i(kz-\omega t)},$$

Then the Navier equation without a volume force, and with constant density,

$$\begin{aligned}
\rho\frac{\partial^2\vec\eta(\vec r)}{\partial t^2} &= -\omega^2\rho\vec A e^{i(kz-\omega t)} \\
&= \left(B+\frac{G}{3}\right)\vec\nabla(\vec\nabla\cdot\vec\eta) + G\nabla^2\vec\eta \\
&= (-k^2\left(B+\frac{G}{3}\right)\hat e_z A_z - k^2 G\vec A)e^{i(kz-\omega t)}
\end{aligned}$$

a) For a longitudinal wave $\vec A = A_z\hat e_z$, and we have a solution provided

$$\frac{\omega}{k} = \sqrt{\frac{B+\frac{4}{3}G}{\rho}} = c_L = 5.2\,\mathrm{km/s}.$$

b) For a transverse wave $A_z = 0$, and we need

$$\frac{\omega}{k} = \sqrt{\frac{G}{\rho}} = c_T = 3.0\,\mathrm{km/s}.$$

c) So if the explosion was a distance D away, the times of travel are D/c_T for the transverse wave and D/c_L for the longitudinal, and if

$$\frac{D}{c_T} - \frac{D}{c_L} = \Delta t, \quad \Longrightarrow D = \Delta t\frac{c_T c_L}{c_L - c_T} = 25\,\mathrm{s}\times\frac{5.2\times3.0}{2.2}\,\mathrm{km/s}$$

or $D = 180$ km.
