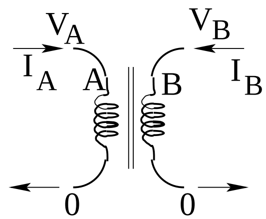
Physics 507 Homework Solution #4
Due: Thursday, Sept. 30, 2010

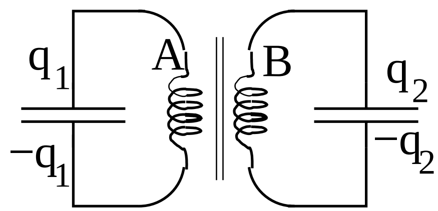
4.1 A transformer consists of two coils of conductor each of which has an inductance, but which also have a coupling, or mutual inductance.

If the current flowing into the upper posts of coils A and B are $I_A(t)$ and $I_B(t)$ respectively, the voltage difference or EMF across each coil is V_A and V_B respectively, where



$$\begin{aligned} V_A &= L_A \frac{dI_A}{dt} + M \frac{dI_B}{dt} \\ V_B &= L_B \frac{dI_B}{dt} + M \frac{dI_A}{dt} \end{aligned}$$

Consider the circuit shown, two capacitors coupled by a such a transformer, where the capacitances are C_A and C_B respectively, with the charges $q_1(t)$ and $q_2(t)$ serving as the generalized coordinates for this problem. Write down the two second order differential equations of "motion" for $q_1(t)$ and $q_2(t)$, and write a Lagrangian for this system.



Solution 4.1: We have defined I_A and I_B in directions so that they represent the rate of *decrease* in the charges q_1 and q_2 respectively. The voltage differences are given by $V_A = q_1/C_A$, $V_B = q_2/C_B$, so

$$\begin{aligned} \frac{q_1}{C_A} &= -L_A \frac{d^2 q_1}{dt^2} - M \frac{d^2 q_2}{dt^2} \\ \frac{q_2}{C_B} &= -L_B \frac{d^2 q_2}{dt^2} - M \frac{d^2 q_1}{dt^2} \end{aligned}$$

Can we get this from a Lagrangian $L = \sum_{ij} \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j - U(q_1, q_2)$, with $m_{ij} = m_{ji}$? That gives equations of motion $\sum_j m_{ij} \ddot{q}_j + \partial U / \partial q_i = 0$, so if

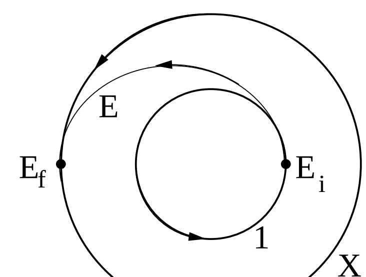
$$m_{ij} = \begin{pmatrix} L_A & M \\ M & L_B \end{pmatrix} \quad \text{and} \quad U = \frac{q_1^2}{2C_A} + \frac{q_2^2}{2C_B},$$

we get the required equations. Note that this is just what one would expect, with the potential energy U given by the usual expression for the energy stored in capacitors, and the "kinetic energy" consisting of not just the self-inductors $\frac{1}{2} L_A I_A^2 + \frac{1}{2} L_B I_B^2$, but also the interaction of the overlapping magnetic fields, $M I_A I_B$.

4.2 A space ship is in circular orbit at radius R and speed v_1, with the period of revolution τ_1. The crew wishes to go to planet X, which is in a circular orbit of radius $2R$, and to revolve around the Sun staying near planet X. They propose to do this by firing two blasts, one putting them in an orbit with perigee R and apogee $2R$, and the second, when near X, to change their velocity so they will have the same speed as X.

- (a) By how much must the first blast change their velocity? Express your answer in terms of v_1.
- (b) How long will it take until they reach the apogee? Express your answer in terms of τ_1
- (c) By how much must the second blast change their speed? Will they need to slow down or speed up, relative to the sun.

Solution 4.2: For motion about a fixed star, the period is proportional to $a^{3/2}$, and the total energy $\propto a^{-1}$, where a is the semimajor axis. For the circular paths, the virial theorem tells us $E = -T = V/2 = -K/2a$. The initial circular path (1), the elliptical path of transit (E), and the final circular path (X) have semimajor axes of $a_1 = R$, $a_E = \frac{3}{2}R$, and $a_X = 2R$ respectively, so they have total energies $E_1 = -K/2R$, $E_E = -K/3R$, $E_X = -K/4R$ respectively, and periods τ_1, $\tau_E = (3/2)^{3/2}\tau_1$, $\tau_X = 2^{3/2}\tau_1$ respectively.



The circular initial and final *kinetic* energies are $T_1 = K/2R$ and $T_X = K/4R$, with potential energies $U_1 = -K/R$ and $U_X = -K/2R$ respectively.

Immediately after the first blast, the potential energy is unchanged at $-K/R$ but the total is now $E_X = -K/3R$, so the kinetic T_{Ei} is $2K/3R$, or 4/3 the initial value. Thus the new velocity is $v_{Ei} = \sqrt{4/3}\, v_1$ which means the blast increased their speed by

$$\Delta v_1 = (\sqrt{4/3} - 1) v_1$$

in the forward direction. The trip takes one half orbital period, or

$$\tau_{\text{trip}} = \tau_E/2 = (3/2)^{3/2}\tau_1/2.$$

What is their speed when then reach planet X? One way to find out is from the energy, as their potential energy will be the same as for the circular orbit of X, $U_{Ef} = -K/2R$, while their total energy is still $U_E = -K/3R$, so their kinetic energy is now $T_{Ef} = K/6R$, for a velocity $v_{Ef} = v_1/\sqrt{3}$. Easier is to use conservation of angular momentum, $L_{zE} = Rv_{Ei} = 2Rv_{Ef}$, so $v_{Ef} = \frac{1}{2}v_{Ei} = v_1/\sqrt{3}$. But $v_X = v_1/\sqrt{2}$ as $T_x = \frac{1}{2}T_1$. Thus they need to accelerate forward to increase their velocity by

$$\Delta v_X = (1/\sqrt{2} - 1/\sqrt{3})v_1.$$

4.3 For the Kepler problem we have the relative position tracing out an ellipse. What is the curve traced out by the momentum in momentum space? Show that it is a circle centered at $\vec{L}\times\vec{A}/L^2$, where $\vec{L}$ and $\vec{A}$ are the angular momentum and Runge-Lenz vectors respectively.

Solution 4.3: From $\vec{A} = \vec{p}\times\vec{L} - \mu K\vec{r}/r$, we have

$$\begin{aligned}\vec{L}\times\vec{A} &= \vec{L}\times(\vec{p}\times\vec{L}) - \frac{\mu^2 K}{r}(\vec{r}\times\vec{p})\times\vec{r} \\ &= L^2\vec{p} - \vec{L}(\vec{p}\cdot\vec{L}) - \frac{\mu^2 K}{r}(r^2\vec{p} - \vec{r}(\vec{r}\cdot\vec{p})) \\ &= L^2\vec{p} - \frac{\mu^2 K}{r}(r^2\vec{p} - \vec{r}(\vec{r}\cdot\vec{p}))\end{aligned}$$

as $\vec{p}$ is perpendicular to $\vec{L}$. So

$$p - \frac{\vec{L}\times\vec{A}}{L^2} = \frac{\mu^2 K}{rL^2}(r^2\vec{p} - \vec{r}(\vec{r}\cdot\vec{p}))$$

and

$$\left(p - \frac{\vec{L}\times\vec{A}}{L^2}\right)^2 = \frac{\mu^4 K^2}{r^2 L^4}(r^4 p^2 - r^2(\vec{r}\cdot\vec{p})^2) = \frac{\mu^4 K^2}{L^4}(\vec{r}\times\vec{p})^2 = \frac{\mu^2 K^2}{L^2},$$

which is a constant. Of course $\vec{p}$ is confined to the plane perpendicular to $\vec{L}$, so the path is a circle of radius $\mu K/|L|$ centered at $\vec{L}\times\vec{A}/L^2$.

4.4 The Rutherford cross section implies all incident projectiles will be scattered and emerge at some angle θ, but a real planet has a finite radius, and a projectile that hits the surface is likely to be captured rather than scattered.

What is the capture cross section for an airless planet of radius R and mass M for a projectile with a speed v_0? How is the scattering differential cross section modified from the Rutherford prediction?

Solution 4.4: The projectile will be captured if the perigee $r_p < R$. The initial energy and angular momentum are $E = \frac{1}{2}mv_0^2$ and $L = mbv_0$. At the perigee r_p, the velocity v_p is perpendicular to the radius, so $L = mr_pv_p$ and $E = \frac{1}{2}mv_p^2 - GMm/r_p$. Thus $v_p = bv_0/r_p$ and $\frac{1}{2}mv_0^2 = \frac{1}{2}mb^2v_0^2/r_p^2 - GMm/r_p$. The impact parameter b_0 corresponding to a perigee of R is given by $b_0 = R\sqrt{1 + \frac{2GM}{Rv_0^2}}$, and the total capture cross section

$$\sigma_{\text{cap}} = \pi b_0^2 = \pi R^2\left(1 + \frac{2GM}{Rv_0^2}\right).$$

The scattering will be unchanged from Rutherford for impact parameters $b > b_0$ corresponding to angles less than the angle θ_0 corresponding to that impact parameter, that is,

$$\tan\frac{\theta_0}{2} = \frac{GM}{Rv_0^2}\frac{1}{\sqrt{1 + \frac{2GM}{Rv_0^2}}}.$$

There will be no scattering for angles larger than θ_0.
