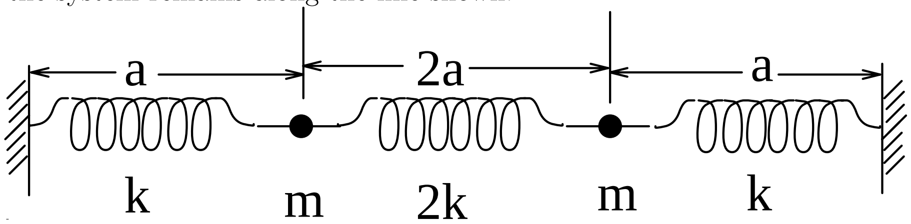
Physics 507 Homework Solutions #7
Due: Thursday, Oct. 21, 2010

7.1 Three springs connect two masses to each other and to immobile walls, as shown. Find the normal modes and frequencies of oscillation, assuming the system remains along the line shown.



Solution 7.1: With x_i measured from the left wall, and with $\eta_1 = x_1 - a$, $\eta_2 = x_2 - 3a$ the displacements from equilibrium, the kinetic and potential energies are

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) \\
V &= \frac{1}{2}k[(x_1 - a)^2 + 2(x_2 - x_1 - 2a)^2 + (x_2 - 3a)^2] \\
&= \frac{1}{2}k[(\eta_1)^2 + (\eta_2)^2 + 2(\eta_2 - \eta_1)^2] \\
&= \frac{1}{2}k[3(\eta_1)^2 + 3(\eta_2)^2 - 4\eta_1\eta_2].
\end{aligned}$$

The kinetic energy is already in the form where the mass matrix is a constant multiple of the identity, so all that is necessary is to diagonalize the potential

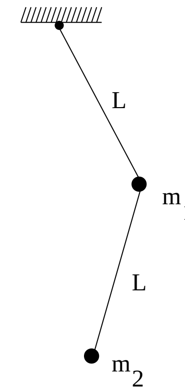
$$B_{jk} = k\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} = \mathcal{O}^{-1}C\mathcal{O},$$

where $\mathcal{O}$ is a rotation by $\pi/4$,

$$\mathcal{O} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad C = k\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

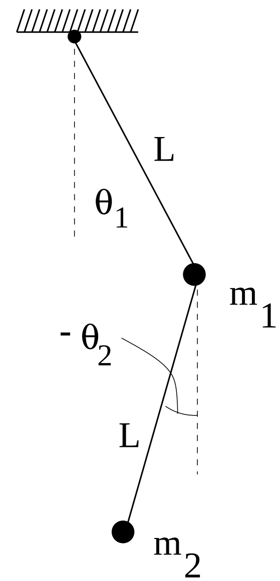
Thus the eigenfrequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{5k/m}$, with corresponding normal modes $\zeta_j =$ Re $(A_j e^{i\omega_j t})$. For the displacements η_j, the first of these corresponds to $\eta = \mathcal{O}^{-1}\zeta_1 = (1,1) \times$ Re $(A_j e^{i\sqrt{k/mt}})$, with both masses moving in the same direction. This corresponds to a body of mass $2m$ and a spring constant $2k$ from the two springs in parallel. The second mode is $(1,-1) \times$ Re $(A_j e^{i\sqrt{5k/mt}})$, with each mass effectively coupled to a fixed point with one spring with constant k and another (half of the middle spring) with constant $4k$.

7.2 Consider the motion, in a fixed vertical plane, of a double pendulum consisting of two masses attached to each other and to a fixed point by inextensible strings of length L. The upper mass has mass m_1 and the lower mass m_2. This is all in a laboratory with the ordinary gravitational forces near the surface of the Earth.



a) Set up the Lagrangian for the motion, assuming the strings stay taut.
b) Simplify the system under the approximation that the motion involves only small deviations from equilibrium. Put the problem in matrix form appropriate for the procedure discussed in class.
c) Find the frequencies of the normal modes of oscillation. [Hint: following exactly the steps given in class will be complex, but the analogous procedure reversing the order of U and T will work easily.]

Solution 7.2: The first mass has coordinates



$$\begin{aligned}
x_1 &= \ell\sin\theta_1 \\
y_1 &= -\ell\cos\theta_1 \\
x_2 &= \ell\,(\sin\theta_1 + \sin\theta_2) \\
y_2 &= -\ell\,(\cos\theta_1 + \cos\theta_2)
\end{aligned}$$

so $v_1^2 = \ell^2\dot{\theta}^2$ and

$$\begin{aligned}
v_2^2 &= \ell^2[(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2)^2 \\
&\quad +(\dot{\theta}_1\sin\theta_1 + \dot{\theta}_2\sin\theta_2)^2] \\
&= \ell^2[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)].
\end{aligned}$$

The potential energy is

$$\begin{aligned}
U &= m_1 g y_1 + m_2 g y_2 \\
&= -g\ell\,(m_1\cos\theta_1 + m_2(\cos\theta_1 + \cos\theta_2))\,.
\end{aligned}$$

For shorthand, let $\rho = m_1/m_2$. Then the Lagrangian is $L = T - U$, where

$$\begin{aligned} T &= \frac{1}{2} m_2 \ell^2 \left[(1+\rho)\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2 \cos(\theta_1 - \theta_2) \right], \\ U &= -m_2 \ell^2 \left[(1+\rho)\cos\theta_1 + \cos\theta_2 \right] \end{aligned}$$

b) Equilbrium is at $\theta_1 = \theta_2 = 0$, so to second order in the θ's and $\dot\theta$'s,

$$\begin{aligned} T &= \frac{1}{2} m_2 \ell^2 \left[(1+\rho)\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2 \right], \\ U &= \frac{1}{2} m_2 \ell^2 \left[(1+\rho)\theta_1^2 + \theta_2^2 \right], \end{aligned}$$

where an irrelevant constant term in the potential energy has been dropped. In matrix form this gives

$$T = \frac{1}{2} m_2 \ell^2 \dot\theta^T \cdot M \cdot \dot\theta, \qquad U = \frac{1}{2} m_2 g \ell \theta^T \cdot A \cdot \theta,$$

with

$$M = \begin{pmatrix} 1+r & 1 \\ 1 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1+r & 0 \\ 0 & 1 \end{pmatrix}.$$

c) As U is already in diagonal form, it is easier if we reverse the steps of the procedure in the book. With $y_1 = \sqrt{1+r}\,\theta_1$ and $y_2 = \theta_2$, we have $U = \frac{1}{2} m_2 g \ell y^T \cdot y$, and $T = \frac{1}{2} \dot y^T \cdot m \cdot \dot y$ with

$$m = \begin{pmatrix} 1 & (1+r)^{-1/2} \\ (1+r)^{-1/2} & 1 \end{pmatrix}.$$

This has equal diagonal elements, so we know the diagonalizing rotation is through 45°, or

$$\begin{aligned} u_1 &= (y_1 + y_2)/\sqrt{2} = (\sqrt{1+r}\,\theta_1 + \theta_2)/\sqrt{2} \\ u_2 &= (y_1 - y_2)/\sqrt{2} = (\sqrt{1+r}\,\theta_1 - \theta_2)/\sqrt{2}, \text{ so} \\ \theta_1 &= (u_1 - u_2)/\sqrt{2}, \\ \theta_2 &= (u_1 + u_2)/\sqrt{2(1+r)} \end{aligned}$$

Then $U = \frac{1}{2} m_2 g \ell (u_1^2 + u_2^2)$ and

$$\begin{aligned} T &= \frac{1}{2} m_2 \ell^2 \left[\frac{1}{2}(\dot u_1 + \dot u_2)^2 + \frac{1}{2}(\dot u_1 - \dot u_2)^2 + \frac{1}{\sqrt{1+r}} (\dot u_1^2 - \dot u_2^2) \right] \\ &= \frac{1}{2} m_2 \ell^2 \left[\left(1 + \frac{1}{\sqrt{1+r}}\right) \dot u_1^2 + \left(1 - \frac{1}{\sqrt{1+r}}\right) \dot u_2^2 \right]. \end{aligned}$$

We now have independent oscillators, with angular frequencies given by the square root of the ration of the coefficient in the potential to that in the kinetic energy. Thus the two frequencies are

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{\ell}} \left(1 \pm \frac{1}{\sqrt{1+r}}\right)^{-1/2}.$$

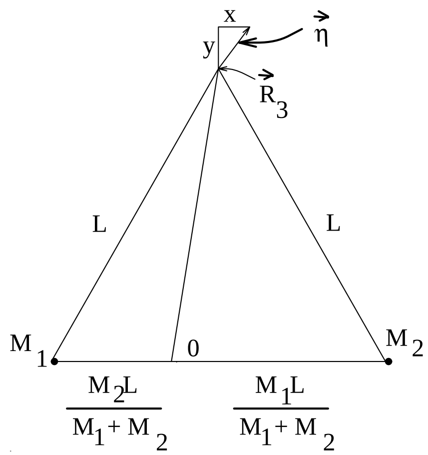
7.3 (a) Show that if three mutually gravitating point masses are at the vertices of an equilateral triangle which is rotating about an axis normal to the plane of the triangle and through the center of mass, at a suitable angular velocity ω, this motion satisfies the equations of motion. Thus this configuration is an equilibrium in the rotating coordinate system. Do not assume the masses are equal.
(b) Suppose that two stars of masses M_1 and M_2 are rotating in circular orbits about their common center of mass. Consider a small mass m which is approximately in the equilibrium position described above (which is known as the L_5 point). The mass is small enough that you can ignore its effect on the two stars. Analyze the motion, considering specifically the stability of the equilibrium point as a function of the ratio of the masses of the stars.

Solution 7.3: If the three particles, which have masses M_i and positions $\vec R_i$, $i = 1, 2, 3$, are in an equilateral triangle of side L, then $|R_i - R_j| = L$ for $i \neq j$. Then the force on particle i is

$$\begin{aligned} \vec F_i &= G M_i \sum_{j \neq i} \frac{M_j (\vec R_j - \vec R_i)}{|\vec R_j - \vec R_i|^3} = G M_i \sum_{j \neq i} \frac{M_j (\vec R_j - \vec R_i)}{L^3} \\ &= G M_i \sum_{\text{all } i} \frac{M_j (\vec R_j - \vec R_i)}{L^3} = \frac{G M_i M_T}{L^3} (\vec R - \vec R_i), \end{aligned}$$

where $M_T = \sum M_i$ is the total mass and $\vec R$ is the position of the center of mass. If the particles are rotating about $\vec R$ with angular velocity ω, the centripetal force required to maintain this motion is $M_i \omega^2 (\vec R - \vec R_i)$, so if $\omega^2 = G M_T L^{-3}$, this is exactly what the gravitational forces do, and the rotation satisfies the equations of motion.

(b) If the third mass $M_3 = m$ is much smaller than the other two, so that its effect on the motion of the others is negligible, M_1 and M_2 will circle their common center of mass at distances $M_2L/(M_1 + M_2)$ and $M_1L/(M_1 + M_2)$ respectively. We choose rotating cartesian coordinates with the origin and z-axis at the center of mass and along $\vec{\omega}$ respectively. From the general equation for the acceleration in a rotating reference frame, we have



$$m\ddot{\vec{r}} = \vec{F} - 2m\vec{\omega}\times\vec{v} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}),$$

If $\vec{r}$ differs from the equilibrium position $\vec{R}_3$ by a vector $\vec{\eta}$, the force on m is

$$\begin{aligned}
\vec{F} &= Gm\left[\frac{M_1\left(\vec{R}_1-\vec{R}_3-\vec{\eta}\right)}{|\vec{R}_1-\vec{R}_3-\vec{\eta}|^3}+\frac{M_2\left(\vec{R}_2-\vec{R}_3-\vec{\eta}\right)}{|\vec{R}_2-\vec{R}_3-\vec{\eta}|^3}\right]\\
&\approx Gm\left[-\frac{M_T}{L^3}\vec{\eta}+M_1\frac{\left(\vec{R}_1-\vec{R}_3\right)}{L^3}+M_2\frac{\left(\vec{R}_2-\vec{R}_3\right)}{L^3}\right. \qquad (1)\\
&\quad\left.+3\,\frac{M_1\left(\vec{R}_1-\vec{R}_3\right)}{L^5}\vec{\eta}\cdot\left(\vec{R}_1-\vec{R}_3\right)+3\frac{M_2\left(\vec{R}_2-\vec{R}_3\right)}{L^5}\vec{\eta}\cdot\left(\vec{R}_2-\vec{R}_3\right)\right],
\end{aligned}$$

where we have expanded to first order in $\vec{\eta}$, using

$$\begin{aligned}
|\vec{R}_1-\vec{R}_3-\vec{\eta}|^{-3} &\approx \left[\left(\vec{R}_1-\vec{R}_3\right)^2-2\vec{\eta}\cdot\left(\vec{R}_1-\vec{R}_3\right)\right]^{-3/2}\\
&\approx L^{-3}\left[1+3\frac{\vec{\eta}\cdot\left(\vec{R}_1-\vec{R}_3\right)}{L^2}\right].
\end{aligned}$$

The second and third terms of (1), which are independent of $\vec{\eta}$, add as at the equilibrium point to $-GmM_T\vec{R}_3/L^3 = -m\omega^2\vec{R}_3$, where we have made use of $\vec{R} = 0$. Let $\vec{\eta} = (x, y, z)$. Then

$$\begin{aligned}
&2\vec{\eta}\cdot\left(\vec{R}_1-\vec{R}_3\right)/L = -x-\sqrt{3}y, \qquad && 2\vec{\eta}\cdot\left(\vec{R}_2-\vec{R}_3\right)/L = x-\sqrt{3}y,\\
&2\left(\vec{R}_1-\vec{R}_3\right)/L = (-1,-\sqrt{3},0), && 2\left(\vec{R}_2-\vec{R}_3\right)/L = (1,-\sqrt{3},0),
\end{aligned}$$

so the last two terms in (1) are

$$\frac{3}{4}\frac{GmM_T}{L^3}\begin{pmatrix} x+\sqrt{3}\xi y\\ \sqrt{3}\xi x+3y\end{pmatrix} = \frac{3}{4}\omega^2\begin{pmatrix} x+\sqrt{3}\xi y\\ \sqrt{3}\xi x+3y\end{pmatrix},$$

where $\xi = (M_1 - M_2)/(M_1 + M_2)$.

Thus

$$\begin{aligned}
\ddot{\vec{\eta}} &= -\omega^2\vec{\eta}-\omega^2\vec{R}_3+\frac{3}{4}\omega^2\begin{pmatrix} x+\sqrt{3}\xi y\\ \sqrt{3}\xi x+3y\\ 0\end{pmatrix}-2\vec{\omega}\times\vec{v}-\vec{\omega}\times\left(\vec{\omega}\times(\vec{\eta}+\vec{R}_3)\right)\\
&= -\omega^2\begin{pmatrix}0\\0\\z\end{pmatrix}+2\omega\begin{pmatrix}\dot{y}\\-\dot{x}\\0\end{pmatrix}+\frac{3}{4}\omega^2\begin{pmatrix} x+\sqrt{3}\xi y\\ \sqrt{3}\xi x+3y\\ 0\end{pmatrix}.
\end{aligned}$$

The z motion decouples simply: $\ddot{z} = -\omega^2 z$, which is stable oscillatory motion. The x and y coordinates have coupled second order linear differential equations. A trial solution $x = ae^{i\alpha\omega t}$, $y = be^{i\alpha\omega t}$ will satisfy the equations if

$$\begin{aligned}
-a\alpha^2 &= 2ib\alpha+\frac{3}{4}(a+\sqrt{3}\xi b),\\
-b\alpha^2 &= -2ia\alpha+\frac{3}{4}(\sqrt{3}\xi a+3b).
\end{aligned}$$

A nonzero solution for (a, b) requires a vanishing determinant:

$$\begin{aligned}
0 &= \det\begin{pmatrix}\alpha^2+3/4 & 2i\alpha+3\sqrt{3}\xi/4\\ -2i\alpha+3\sqrt{3}\xi/4 & \alpha^2+9/4\end{pmatrix}\\
&= \left(\alpha^2+3/4\right)\left(\alpha^2+9/4\right)-\left(\frac{27}{16}\xi^2+4\alpha^2\right)\\
&= \alpha^4-\alpha^2+\frac{27}{16}\left(1-\xi^2\right)
\end{aligned}$$

The requirement for a stable solution is that all solutions have $\operatorname{Im}\alpha \geq 0$, but this can only be true if $2\alpha^2 = 1\pm\sqrt{1-27(1-\xi^2)/4}$ is real and positive. As $\xi \in [-1, 1]$ by its definition, this requires only that the argument of the square root is nonnegative, or $\xi^2 \geq 23/27$. Calling M_1 the greater of the two masses, we have $\xi > 0$, so $.923 \leq \xi = 1 - 2M_2/(M_1 + M_2)$, or $(M_1 + M_2)/M_2 \geq (.0385)^{-1} = 25.96$, or

$$\frac{M_1}{M_2} \geq 24.96$$

is required for a stable equilibrium. In that case, we have shown that the linear equations have stable solutions. We are still not sure of absolute stability, as higher order considerations might cause a slow deviation from this oscillatory solution, and this problem was not solved until 1962[1].

The mass of the Earth is 81 times the mass of the moon, so there are two points, called L_4 and L_5, after Lagrange, who discovered them, which would be stable places to store things. This has been proposed for space colonies. The Sun is about 1000 times more massive than Jupiter, so if the effects of the other planets can be ignored, there should be stable points for the Jupiter–Sun system, and indeed there are accumulations of asteroids, called the Trojan asteroids, at both these points.

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[1] J. K. Moser, *Lectures on Hamilitonian Systems*, Memoirs of the Amer. Math. Soc., **81**, 1968