Physics 507 Homework Solution #9
Due: Nov. 11, 2010

9.1 Two lagrangians, L_1 and L_2, which differ by a total time derivative of a function on extended configuration space,

$$L_1(\{q_i\},\{\dot q_j\},t) = L_2(\{q_i\},\{\dot q_j\},t) + \frac{d}{dt}\Phi(q_1,...,q_n,t),$$

describe the same dynamics. That is, they give the same equations of motion $q_i(t)$, but they give differing momenta $p_i^{(1)}$ and $p_i^{(2)}$. Find the relationship between the two momenta and between the two Hamiltonians, H_1 and H_2, and show that these Hamiltonians lead to equivalent equations of motion.

Solution 9.1: If Φ is a function of q_i and t, a total time derivative is

$$\frac{d\Phi}{dt} = \sum_i \frac{\partial\Phi}{\partial q_i}\dot q_i + \frac{\partial\Phi}{\partial t}.$$

If we have an original Lagrangian L_2 with momenta $p_i^{(2)}$ and Hamiltonian H_2, defining a new Lagrangian

$$L_1 = L_2 + \sum_i \frac{\partial\Phi}{\partial q_i}\dot q_i + \frac{\partial\Phi}{\partial t}$$

leads to new momenta

$$p_i^{(1)} = \frac{\partial L_1}{\partial \dot q_i} = \frac{\partial L_2}{\partial \dot q_i} + \frac{\partial\Phi}{\partial q_i} = p_i^{(2)} + \frac{\partial\Phi}{\partial q_i}$$

and a new Hamiltonian

$$\begin{aligned} H_1 &= \sum_i p_i^{(1)}\dot q_i - L_1 = \sum_i\left(p_i^{(2)}\dot q_i + \frac{\partial\Phi}{\partial q_i}\dot q_i\right) - L_2 - \sum_i \frac{\partial\Phi}{\partial q_i}\dot q_i - \frac{\partial\Phi}{\partial t} \\ &= H_2 - \frac{\partial\Phi}{\partial t}. \end{aligned}$$

The new equations of motion are

$$\begin{aligned} \dot q_i &= \left.\frac{\partial H_1}{\partial p_i^{(1)}}\right|_q = \sum_j \left.\frac{\partial H_2}{\partial p_j^{(2)}}\right|_q \left.\frac{\partial p_j^{(2)}}{\partial p_i^{(1)}}\right|_q = \left.\frac{\partial H_2}{\partial p_i^{(2)}}\right|_q \\ \dot p_i^{(1)} &= -\left.\frac{\partial H_1}{\partial q_i}\right|_{p^{(1)}} = -\left.\frac{\partial H_2}{\partial q_i}\right|_{p^{(1)}} + \frac{\partial^2\Phi}{\partial q_i\partial t}. \end{aligned}$$

We need to evaluate by the chain rule

$$\begin{aligned} \left.\frac{\partial H_2}{\partial q_i}\right|_{p^{(1)}} &= \sum_j \left.\frac{\partial H_2}{\partial q_j}\right|_{p^{(2)}} \left.\frac{\partial q_j}{\partial q_i}\right|_{p^{(1)}} + \sum_j \left.\frac{\partial H_2}{\partial p_j^{(2)}}\right|_q \left.\frac{\partial p_j^{(2)}}{\partial q_i}\right|_{p^{(1)}} \\ &= \sum_j \left.\frac{\partial H_2}{\partial q_j}\right|_{p^{(2)}} \delta_{ij} + \sum_j \left.\frac{\partial H_2}{\partial p_j^{(2)}}\right|_q \left.\frac{\partial p_j^{(2)}}{\partial q_i}\right|_{p^{(1)}} \\ &= \left.\frac{\partial H_2}{\partial q_i}\right|_{p^{(2)}} - \sum_j \dot q_j \frac{\partial^2\Phi}{\partial q_i\partial q_j}. \end{aligned}$$

Thus

$$\dot p_i^{(1)} = -\left.\frac{\partial H_2}{\partial q_i}\right|_{p^{(2)}} + \sum_j \dot q_j \frac{\partial^2\Phi}{\partial q_i\partial q_j} + \frac{\partial^2\Phi}{\partial q_i\partial t} = -\left.\frac{\partial H_2}{\partial q_i}\right|_{p^{(2)}} + \frac{d}{dt}\frac{\partial\Phi}{\partial q_i}.$$

Then

$$\dot p_i^{(2)} = \dot p_i^{(1)} - \frac{d}{dt}\frac{\partial\Phi}{\partial q_i} = -\left.\frac{\partial H_2}{\partial q_i}\right|_{p^{(2)}},$$

agreeing with what the equation of motion from the original Hamiltonian.

9.2 A uniform static magnetic field can be described by a static vector potential $\vec A = \frac{1}{2}\vec B\times\vec r$. A particle of mass m and charge q moves under the influence of this field.
(a) Find the Hamiltonian, using inertial cartesian coordinates.
(b) Find the Hamiltonian, using coordinates of a rotating system with angular velocity $\vec\omega = -q\vec B/2mc$.

Solution 9.2: a.) At the end of Chapter 2, we found the velocity dependent potential which describes an electromagnetic field to be

$$U_{\text{em}} = q\left(\phi(r,t) - \vec{v}\cdot\vec{A}(\vec{r},t)/c\right).$$

Here we have $\phi \equiv 0$ and $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$. We also have an additional nonelectromagnetic term $U(\vec{r})$. Thus

$$L(\vec{r},\vec{v},t) = \frac{1}{2}m\vec{v}^2 + \frac{q}{2c}(\vec{B}\times\vec{r})\cdot\vec{v} - U(\vec{r}).$$

The momentum is

$$\vec{p} = m\vec{v} + \frac{q}{2c}\vec{B}\times\vec{r},$$

so the Hamiltonian is

$$\begin{aligned} H &= \vec{v}\cdot\vec{p} - L = \frac{1}{2}m\vec{v}^2 + U(\vec{r}) \\ &= \frac{p^2}{2m} - \frac{q}{2mc}\vec{p}\cdot\left(\vec{B}\times\vec{r}\right) + \frac{q^2}{8mc^2}(\vec{B}\times\vec{r})^2 + U(\vec{r}). \end{aligned}$$

(b.) In terms of a rotating coordinate system,

$$\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_b + \vec{\omega}\times\vec{r},$$

so the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)_b^2 + m\left(\frac{d\vec{r}}{dt}\right)_b\cdot(\vec{\omega}\times\vec{r}) + \frac{1}{2}m(\vec{\omega}\times\vec{r})^2 \\ &\quad + \frac{q}{2c}(\vec{B}\times\vec{r})\cdot\left(\frac{d\vec{r}}{dt}\right)_b + \frac{q}{2c}(\vec{B}\times\vec{r})\cdot(\vec{\omega}\times\vec{r}) - U(\vec{r}) \\ &= \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)_b^2 + m\left(\frac{d\vec{r}}{dt}\right)_b\cdot\left(\left[\vec{\omega} + \frac{q}{2mc}\vec{B}\right]\times\vec{r}\right) \\ &\quad + \frac{1}{2}m(\vec{\omega}\times\vec{r})\cdot\left(\left[\vec{\omega} + \frac{q}{mc}\vec{B}\right]\times\vec{r}\right) - U(\vec{r}) \end{aligned}$$

If $\vec{\omega} = -q\vec{B}/2mc$, the term linear in $\dot{r}$ vanishes, and we have

$$L = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)_b^2 - \frac{q^2}{8mc^2}(\vec{B}\times\vec{r})^2 - U(\vec{r}).$$

Then

$$\vec{p} = m\dot{\vec{r}}, \qquad H = \frac{p^2}{2m} + \frac{q^2}{8mc^2}(\vec{B}\times\vec{r})^2 + U(\vec{r}).$$

Notice that we now have a Hamiltonian of the most usual kind, with a non-velocity-dependent potential,

$$U'(\vec{r}) = U(\vec{r}) + \frac{q^2}{8mc^2}\left[B^2r^2 - (\vec{B}\cdot\vec{r})^2\right] = U(\vec{r}) + \frac{q^2B^2}{8mc^2}r_\perp^2.$$

9.3 (a) Show directly that the transformation

$$Q = \ln\left(\frac{\sin p}{q}\right), \qquad P = q\cot p$$

is canonical.

(b) Show directly that, for a arbitrary fixed constant α,

$$Q = \arctan\left(\frac{\alpha q}{p}\right), \qquad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical.

Solution: (a) From

$$Q = \ln\left(\frac{\sin p}{q}\right), \qquad P = q\cot p$$

we need the partial derivatives

$$\frac{\partial Q}{\partial q} = \frac{1}{\frac{1}{q}\sin p}\frac{-1}{q^2}\sin p = -\frac{1}{q}, \qquad \frac{\partial Q}{\partial p} = \frac{1}{\frac{1}{q}\sin p}\frac{\cos p}{q} = \cot p,$$
$$\frac{\partial P}{\partial q} = \cot p, \qquad \frac{\partial P}{\partial p} = -q\csc^2 p$$

With $\eta = \begin{pmatrix} q \\ p \end{pmatrix}$ and $\zeta = \begin{pmatrix} Q \\ P \end{pmatrix}$,

$$M = \begin{pmatrix} -1/q & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix},$$

So[1]

$$
\begin{aligned}
M\cdot J\cdot M^T &= \begin{pmatrix} -1/q & \cot p \\ \cot p & -q\csc^2 p\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} -1/q & \cot p \\ \cot p & -q\csc^2 p\end{pmatrix} \\
&= \begin{pmatrix} -1/q & \cot p \\ \cot p & -q\csc^2 p\end{pmatrix}\begin{pmatrix} \cot p & -q\csc^2 p \\ 1/q & -\cot p\end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{q}\cot p + \frac{1}{q}\cot p & \csc^2 p - \cot^2 p \\ \cot^2 p - \csc^2 p & -q\cot p\csc^2 p + q\cot p\csc^2 p\end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = J
\end{aligned}
$$

so this transformation satisfies the necessary condition, $M\cdot J\cdot M^T = J$, to be canonical.

(b) With

$$
Q = \arctan\left(\frac{\alpha q}{p}\right), \qquad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)
$$

we have

$$
\begin{aligned}
\frac{\partial Q}{\partial q} &= \frac{\alpha}{p}\frac{1}{1+\frac{\alpha^2 q^2}{p^2}} = \frac{\alpha p}{p^2+\alpha^2 q^2} \\
\frac{\partial Q}{\partial p} &= -\frac{\alpha q}{p^2}\frac{1}{1+\frac{\alpha^2 q^2}{p^2}} = -\frac{\alpha q}{p^2+\alpha^2 q^2}
\end{aligned}
$$

$$
\frac{\partial P}{\partial q} = \alpha q, \qquad \frac{\partial P}{\partial p} = \frac{p}{\alpha}
$$

So[2]

$$
M = \begin{pmatrix} \dfrac{\alpha p}{p^2+\alpha^2q^2} & -\dfrac{\alpha q}{p^2+\alpha^2 q^2} \\ \alpha q & \dfrac{p}{\alpha}\end{pmatrix}
$$

$$
\begin{aligned}
M\cdot J\cdot M^T &= \begin{pmatrix} \dfrac{\alpha p}{p^2+\alpha^2q^2} & -\dfrac{\alpha q}{p^2+\alpha^2 q^2} \\ \alpha q & \dfrac{p}{\alpha}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} \dfrac{\alpha p}{p^2+\alpha^2q^2} & \alpha q \\ -\dfrac{\alpha q}{p^2+\alpha^2 q^2} & \dfrac{p}{\alpha}\end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\alpha p}{p^2+\alpha^2q^2} & -\dfrac{\alpha q}{p^2+\alpha^2 q^2} \\ \alpha q & \dfrac{p}{\alpha}\end{pmatrix}\begin{pmatrix} \dfrac{-\alpha q}{p^2+\alpha^2q^2} & \dfrac{p}{\alpha} \\ -\dfrac{\alpha p}{p^2+\alpha^2 q^2} & -\alpha q\end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\alpha^2 pq - \alpha^2 pq}{(p^2+\alpha^2q^2)^2} & \dfrac{p^2+\alpha^2q^2}{(p^2+\alpha^2 q^2)} \\ \dfrac{-\alpha^2q^2+p^2}{(p^2+\alpha^2q^2)} & pq - pq\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = J.
\end{aligned}
$$

[1]See alternative short cut at end of (b).

[2]Again, see short cut below

Short cut for single degree of freedom

In general we need to check $M\cdot J\cdot M^T = J$, but that is very easy for a two dimensional M. Because both sides are automatically antisymmetric, there is just one equation on the matrix elements. With

$$
\begin{aligned}
M &= \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \quad M\cdot J\cdot M^T = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} a & c \\ b & d\end{pmatrix} \\
&= (ad-bc)\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix},
\end{aligned}
$$

so all we need to check is that $\det M = ad - bc = 1$.

For more degrees of freedom, however, $\det M = 1$ is not sufficient.