Physics 507 Homework Solutions #11
Due: Thursday, Dec. 2, 2010

11.1 (a) Show that a particle under a central force with an attractive *potential* inversely proportional to the distance *squared* has a conserved quantity $D = \frac{1}{2}\vec{r}\cdot\vec{p} - Ht$.
(b) Show that the infinitesimal transformation generated by $G := \frac{1}{2}\vec{r}\cdot\vec{p}$ scales $\vec{r}$ and $\vec{p}$ by opposite infinitesimal amounts, $\vec{Q} = (1+\frac{\epsilon}{2})\vec{r}$, $\vec{P} = (1-\frac{\epsilon}{2})\vec{p}$, or for a finite transformation $\vec{Q} = \lambda\vec{r}$, $\vec{P} = \lambda^{-1}\vec{p}$. Show that if we describe the motion in terms of a scaled time $T = \lambda^2 t$, the equations of motion are invariant under this combined transformation $(\vec{r},\vec{p},t) \to (\vec{Q},\vec{P},T)$.

Solution 11.1: A particle under a central force with an attractive potential inversely proportional to the distance *squared* has a Hamiltonian $H = p^2/2m - k/r^2$. The quantity $D = \frac{1}{2}\vec{r}\cdot\vec{p} - Ht$ will be conserved if

$$\frac{dD}{dt} \equiv [D,H] + \frac{\partial D}{\partial t} = 0.$$

The last term in D has zero bracket with H, as $[H,H] = 0$ by symmetry, so the Poisson bracket is

$$[D,H] = \sum_i \left(\frac{1}{2}\frac{\partial \vec{r}\cdot\vec{p}}{\partial r_i}\frac{\partial H}{\partial p_i} - \frac{1}{2}\frac{\partial \vec{r}\cdot\vec{p}}{\partial p_i}\frac{\partial H}{\partial r_i}\right) = \frac{1}{2}\sum_i \left(p_i\frac{p_i}{m} - r_i\frac{2kr_i}{r^4}\right) = H,$$

and as $\partial D/\partial t = -H$, D is conserved
(b) Under a infinitesimal transformation generated by ϵG, the phase-space coordinates change by

$$\delta\eta = \epsilon[\eta, G].$$

Here G is $\frac{1}{2}\vec{r}\cdot\vec{p}$. Thus

$$\begin{aligned}\delta r_i &= \epsilon[r_i, \frac{1}{2}\vec{r}\cdot\vec{p}] = \epsilon\frac{\partial}{\partial p_i}\left(\frac{1}{2}\vec{r}\cdot\vec{p}\right) = \frac{1}{2}r_i \\ \delta p_i &= \epsilon[p_i, \frac{1}{2}\vec{r}\cdot\vec{p}] = -\epsilon\frac{\partial}{\partial r_i}\left(\frac{1}{2}\vec{r}\cdot\vec{p}\right) = -\frac{1}{2}p_i\end{aligned}$$

or $\vec{r} \to (1+\epsilon/2)\vec{r}$, $\vec{p} \to (1-\epsilon/2)\vec{p} \sim (1+\epsilon/2)^{-1}\vec{p}$. Under a finite transformation these factors will build up to a finite factor, and $\vec{r} \to \vec{Q} = \lambda\vec{r}$, $\vec{p} \to \vec{P} = \lambda^{-1}\vec{p}$.

The equations of motion in the orginal system, with $H = \frac{1}{2}p^2/2m - k/r^2$, are

$$\dot{p}_i = -\frac{\partial H}{\partial r_i} = \frac{2kr_i}{r^4}, \qquad \dot{r}_i = \frac{\partial H}{\partial p_i} = p_i/m.$$

Let $\vec{Q} = \lambda\vec{r}$, $\vec{P} = \lambda^{-1}\vec{p}$, $t' = \lambda^2 t$, then

$$\begin{aligned}\frac{d\vec{P}}{dt'} &= \lambda^{-2}\dot{\vec{P}} = \lambda^{-3}\dot{\vec{p}} = \frac{2kr_i}{\lambda^3 r^4} = \frac{2kQ_i}{Q^4} \\ \frac{d\vec{Q}}{dt'} &= \lambda^{-2}\dot{\vec{Q}} = \lambda^{-1}\dot{\vec{r}} = \lambda^{-1}\vec{p} = \vec{P},\end{aligned}$$

so the form of the Hamiltonian equations is unchanged.

Another way to see this is to note that, as G has no explicit time dependence, the Hamiltonian is unchanged as a function on phase space, $K(Q,P) = H(r,p)$, but as a function of its arguments,

$$K(Q,P) = H(r,p) = \frac{1}{2m}p^2 - \frac{k}{r^2} = \frac{1}{2m}(\lambda P)^2 - \frac{\lambda^2 k}{Q^2} = \lambda^2 H(Q,P).$$

Thus Q and P obey the same differential equations as r and p, except that the time derivatives are multiplied by λ^2. Thus if $q(t) = f(t), p(t) = g(t)$ is a solution for Hamilton's equations for $H(q,p)$, $Q(t) = f(\lambda^2 t), P(t) = g(\lambda^2 t)$ is a solution for those of $K(Q,P)$.

11.2 Consider a particle of mass m and charge q in the field of a fixed electric dipole with dipole moment[1] p. In spherical coordinates, the potential energy is given by

$$U(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta.$$

a) Write the Hamiltonian. It is independent of t and ϕ. As a consequence, there are two conserved quantities. What are they?
b) Find the partial differential equation in t, r, θ, and ϕ satisfied by Hamilton's principal function S, and the partial differential equation in r, θ, and ϕ satisfied by Hamilton's characteristic function W.
c) Assume W can be broken up into r-dependent, θ-dependent, and ϕ-dependent pieces:

$$W(r,\theta,\phi,P_i) = W_r(r,P_i) + W_\theta(\theta,P_i) + W_\phi(\phi,P_i).$$

Find ordinary differential equations for W_r, W_θ and W_ϕ.

Solution 11.2: a) The Hamiltonian[2]

$$H(r,\theta,\phi,p_r,p_\theta,p_\phi,t) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta$$

[1]Please note that q and p are the charge and dipole moments here, not coordinates or momenta of the particle.
[2]But p_r, p_θ, p_ϕ, P_i are momenta and q_i are coordinates.

is time independant and therefore conserved. Also ϕ does not enter the Hamiltonian, so it is an ignorable coordinate, and therefore p_ϕ is conserved.
b) In general, $S(q_i, P_i, t)$ satisfies[3]

$$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0,$$

so

$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2mr^2 \sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta + \frac{\partial S}{\partial t} = 0.$$

As H is time-independent, we may assume the S can be broken into the time independent Hamilton characteristic function $W(q_i, P_i)$, together with a piece independent of $\{q_i\}$, $-\alpha(P_i)t$, then

$$\begin{aligned} H \quad & \left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha \\ & = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{2mr^2 \sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta. \end{aligned}$$

c) If $W(r, \theta, \phi, P_i) = W_r(r, P_i) + W_\theta(\theta, P_i) + W_\phi(\phi, P_i)$ then

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{1}{2mr^2 \sin^2\theta}\left(\frac{dW_\phi}{d\phi}\right)^2 + \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta = \alpha. \tag{1}$$

Only the third term depends on ϕ, so $dW_\phi/d\phi = \beta$, a constant, so $W_\phi = \beta\phi + a$. Inserting this back into (1), and multiplying by $2mr^2$, we get

$$r^2\left(\frac{dW_r}{dr}\right)^2 - 2m\alpha r^2 + \left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\beta^2}{\sin^2\theta} + \frac{mqp}{2\pi\epsilon_0}\cos\theta = 0.$$

Only the first two terms depend on r, and they depend only on r, so

$$r^2\left(\frac{dW_r}{dr}\right)^2 - 2m\alpha r^2 = \gamma,$$

a constant, and then

$$\left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\beta^2}{\sin^2\theta} + \frac{mqp}{2\pi\epsilon_0}\cos\theta = -\gamma.$$

These last two equations are ordinary differential equations for W_r and W_θ.

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[3] P_i are constants, the coordinates in the canonically transformed coordinates where nothing changes.