Physics 507 Homework Solution #12

1. If we use polar coordinates, the potential energy is $U = -K/r$, the kinetic energy is $T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2)$. The momenta conjugate to r and θ are then $p_r = \mu\dot r$ and $p_\theta = \mu r^2\dot\theta$ respectively, and as the Lagrangian has no explicit time dependence and θ is an ignorable coordinate, the Hamiltonian $H = p_r^2/2m + p_\theta^2/2mr^2 + K/r$ and p_θ are conserved, *i.e.* are integrals of the motion. Note p_θ is the angular momentum about the z-axis, *i.e.* perpendicular to the plane.
a) Thus $F_1 = H$ and $F_2 = p_\theta$ are two integrals of the motion.
b) As θ does not appear in H, $[H, p_\theta] = 0$, and the two are in involution.
c) For a given $\vec f = (E, L)$, the manifold $\mathcal{M}_{\vec f}$ consists of all points in phase space consistent with that E and $p_\theta = L$. As the semi-major axis and ellipticity of the elliptical orbit is determined by E and L, $\mathcal{M}_{\vec f}$ has all points possible for all such ellipses. Thus its projection onto coordinate space is an annulus with perigee and apogee r_p and r_a as the radii. The momentum p_r is given, modulo sign, by $\sqrt{2m(E - U(r))}$, so our invariant torus really is a torus. The remaining momentum, p_θ, is a constant. We may choose η_0 any point on it, so let $\eta_0 = (r_a, 0, 0)$

The other points in $\eta \in \mathcal{M}_{\vec f}$ are generated by the canonical transformation $g^{\vec t}$, *i.e.* $\eta = g^{\vec t}(\eta_0)$. The parameters t_1, t_2 give the parameters by which the generators $F_1 = H$ and $F_2 = p_\theta$ have been applied. The momentum as a generator generates a translation in the conjugate coordinate, so $\eta = (r, \theta, p_r) = g^{0,t_2}(\eta_0) = (r_a, t_2, 0)$, while the generator H moves the phase space point forwards in time according to the standard newtonian laws. Thus if we solved the Kepler problem for $\vec r(0) = r_a\hat e_x$, $p_r(0) = 0$, $p_\theta = L$, the solution $(r(t), \theta(t), p_r(t))$ is the value $\eta = g^{t,0}(\eta_0)$.
d) Clearly $\theta \to \theta + 2\pi$ brings us back to the same point in phase space, so there is a periodicity under $t_2 \to t_2 + 2\pi$. We also know the dynamical motion is periodic with period $T = \pi K\sqrt{\mu/2}(-E)^{-3/2}$, so this is the period of t_1. Thus the $\vec e_i$ are $(0, 2\pi)$ and $(T, 0)$ respectively, A is diagonal with elements $T/2\pi$ and 1, and the frequencies $\omega_i = (A^{-1})_{i1} = (2\pi/T, 0)$.
e) The relation $\omega_2 = 0$ is a relation among the frequencies, which is there independent of the values $\vec f$ of the integrals of the motion, so we have here a degenerate system.

2. Using polar coordinates,

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r),$$

for which θ is an ignorable coordinate and thus $p_\theta = L$ is an integral of the motion (and conserved). Thus $F_1 = H$ and $F_2 = p_\theta$ are the integrals of the motion in involution required for our problem to be an integrable system.

The invariant torus is the region of phase space for which $H(r, \theta, p_r, p_\theta) = E$, $p_\theta = L$. (In terms of the general discussion, $\vec f = (E, L)$.) As θ does not enter these constraints, the torus is the cartesian product of the one-dimensional solution of

$$E = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + U(r)$$

with the unit circle for θ. For negative E the system is bound, and the motion $r(t)$ is periodic, with some period τ_r, with r confined to the region $r \in [r_{\min}, r_{\max}]$. The evolution of θ is also determined simply by $\dot\theta = L/mr^2$. Notice $\dot\theta$ is not constant, but it never changes sign.

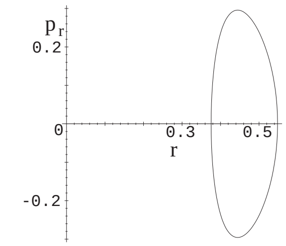

Fig. 1. The cross section of the invariant torus for a given value of θ. This is for $U(r) = -K/r^{3/2}$, with m=1, L=1, K=1, E=$-$0.8.

The action of p_θ on $\mathcal{M}_{\vec f}$ is particularly simple, as $D_{p_\theta} = -\partial/\partial\theta$, and has no effect on r or p_r. D_H, on the other hand, generates the dynamical motion expected from Newton's equations, so that r, p_r, and θ all vary with time t_1. Suppose we chose η_0 to be the point $(r = r_{\max}, p_\theta = 0, \theta = 0)$. The action of $t_1 H$ on this point,

$$g^{t_1 H}(\eta_0) = (r(t_1), p_r(t_1), \theta(t_1)),$$

so given any point $\eta = (r, p_r, \theta)$ on $\mathcal{M}_{\vec f}$, we can find a $t_1 \in [-\tau_r/2, \tau_r/2]$ determined by the magnitude of r and the sign of p_r. Then to make $\eta = g^{t_1 H + t_2 p_\theta}(\eta_0)$, we need only to choose $t_2 = \theta - \theta(t_1)$.

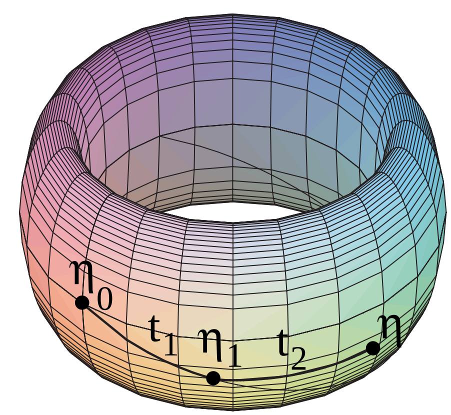

Fig. 2. The invariant torus. The point η_0 flows under the generator $t_1 H$ to η_1, and then under $t_2 p_\theta$ to η.

The generator p_θ clearly brings η_0 back to itself for $t_2 = 2\pi$. The action of H is more complicated — to get back to the same (r, p_r) we need $t_1 = \tau_r$, but that changes θ by some angle $\Delta\theta = \theta(\tau_r)$. This is undone by $t_2 = -\Delta\theta$. Thus the generators of the unit cell are $\vec e_1 = (\tau_r, -\Delta\theta)$ and $\vec e_2 = (0, 2\pi)$.

Then as $2\pi A_{ji} = (\vec{e}_i)_j$,

$$A = \begin{pmatrix} \dfrac{\tau_r}{2\pi} & 0 \\ -\dfrac{\Delta\theta}{2\pi} & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} \dfrac{2\pi}{\tau_r} & 0 \\ \dfrac{\Delta\theta}{\tau_r} & 1 \end{pmatrix}.$$

Defining the ϕ_i by $\vec{t} = \sum_j \phi_j \vec{e}_j / 2\pi$, ($t_i = \sum_j A_{ij}\phi_j$, $\vec{\phi} = A^{-1} \cdot \vec{t}$), so

$$\omega_i = \left(A^{-1}\right)_{i1} = \left(\frac{2\pi}{\tau_r}, \frac{\Delta\theta}{\tau_r}\right).$$

We see in particular that $\delta\phi_2 = \delta t_2 + \Delta\theta \delta t_1/\tau_r$ is not just the δt_2 generated by p_θ, for the Hamiltonian F_1 does make θ flow.

The Hamiltonian produces a flow spiraling around the torus, while p_θ produces only a rotation about the p_r axis. The ratio of the frequencies is $\omega_2/\omega_1 = \Delta\theta/2\pi$, so there will be a relation among the frequencies if $\Delta\theta/2\pi$ is rational. For example, we show the trajectories (of hamiltonian motion) for two different energies, with $\Delta\theta = 0.70597.. \times 2\pi$ and $\Delta\theta = \frac{2}{3} \times 2\pi$.

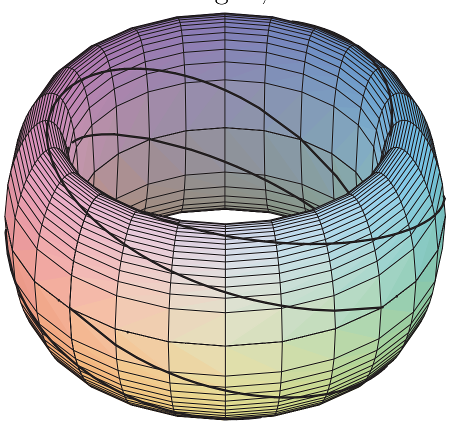

Fig. 3. Hamiltonian flow in the situation of Fig. 1, with $E = -0.8$.

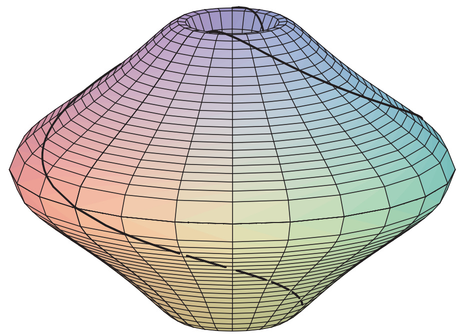

Fig. 4. With the energy changed to $E = -0.24$, $\Delta\theta = 4\pi/3$, so there is a relation among the frequencies with $k_1 = 2, k_2 = -3$.

Each invariant torus corresponds to a given value of E and p_θ. For different E the tori are nested inside one another, as shown. In the case pictured here, corresponding to an attractive central force $F = -Kr^{-5/2}$, the ratio of the periods depends on the energy. In particular, Fig. 3 fits between the the torus with $E = -0.82$ and the torus (half cut away here) with $E = -0.75$, and both are inside of the torus shown in Fig. 4.