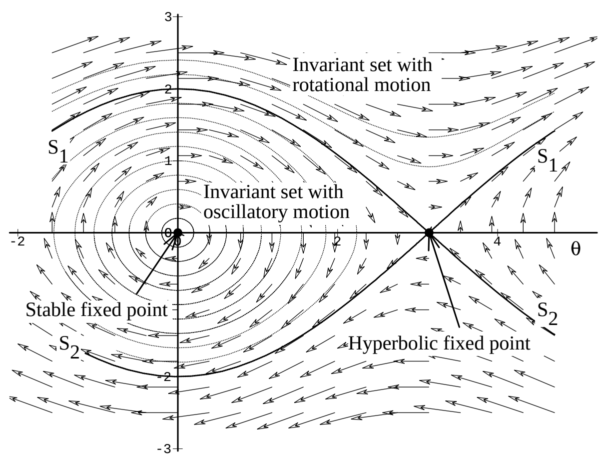
Physics 507 Homework Solution #3
Due: Thursday, Sept. 23, 2010

3.1 Consider a pendulum consisting of a mass at the end of a massless rod of length L, the other end of which is fixed but free to rotate. Ignore one of the horizontal directions, and describe the dynamics in terms of the angle θ between the rod and the downwards direction, without making a small angle approximation.
(a) Find the generalized force Q_θ and find the conserved quantity on phase space.
(b) Give a sketch of the velocity function, including all the regions of phase space. Show all fixed points, separatrices, and describe all the invariant sets of states. [Note: the variable θ is defined only modulo 2π, so the phase space is the Cartesian product of an interval of length 2π in θ with the real line for p_θ. This can be plotted on a strip, with the understanding that the left and right edges are identified. To avoid having important points on the boundary, it would be well to plot this with $\theta \in [-\pi/2, 3\pi/2]$.

Solution 3.1: (a) The Lagrangian is $L = \frac{1}{2}mL^2(\dot\theta)^2 - U$, with $U = -mgL\cos\theta$, so $Q(\theta) = -\partial U/\partial\theta = -mgL\sin\theta$. The conserved quantity is $E = T+U = \frac{1}{2}mL^2(\dot\theta)^2 - mgL\cos\theta$. As $p = \partial L/\partial\dot\theta = mL^2\dot\theta$, the conserved energy is $E(\theta, p) = p^2/2mL^2 - mgL\cos\theta$.

(b) In the figure, the angle is graphed for $\theta \in [-\pi/2, 3\pi/2]$. S_1 and S_2 are separatrices. There is one stable and one unstable fixed point, while the rest of phase space consists of invariant sets with oscillatory motion and invariant sets with rotational motion.



3.2 Consider again the pendulum of mass m on a massless rod of length L, with motion restricted to a fixed vertical plane, with θ, the angle made with the downward direction, the generalized coordinate. Using the fact that the energy E is a constant,
(a) Find $d\theta/dt$ as a function of θ.
(b) Assuming the energy is such that the mass comes to rest at $\theta = \pm\theta_0$, find an integral expression for the period of the pendulum.
(c) Show that the answer is $4\sqrt{\frac{L}{g}}K(\sin^2(\theta_0/2)$, where

$$K(m) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}$$

is the complete elliptic integral of the first kind.
(Note: the circumference of an ellipse is $4aK(e^2)$, where a is the semi-major axis and e the eccentricity.)
(d) Show that $K(m)$ is given by the power series expansion

$$K(m) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 m^n,$$

and give an estimate for the ratio of the period for $\theta_0 = 60°$ to that for small angles.

Solution 3.2: (a) The energy is $E = \frac{1}{2}mL^2\dot\theta^2 - mgL\cos\theta$, where we have chosen $U = 0$ at the pivot height. Then

$$\dot\theta = \sqrt{\frac{2E}{mL^2} + \frac{2g}{L}\cos\theta} = \sqrt{\frac{2g}{L}}\sqrt{\cos\theta - \cos\theta_0},$$

where $gmL\cos\theta_0 = -E$.
(b) Thus the period is

$$T = 2\int_{-\theta_0}^{\theta_0}\sqrt{\frac{L}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,d\theta.$$

(c) As $\cos\theta = 1 - 2\sin^2(\theta/2)$, if we let $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$,

$$\begin{aligned}
T &= 4\sqrt{\frac{L}{2g}}\int_0^{\theta_0} d\theta \frac{1}{\sqrt{2\left(\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})\right)}} \\
&= 2\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{2\sin(\theta_0/2)\cos\phi d\phi}{\cos(\theta/2)}\frac{1}{\sin(\theta_0/2)}\frac{1}{\sqrt{1-\sin^2\phi}} \\
&= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2\phi}} = 4\sqrt{\frac{L}{g}}K(\sin^2(\theta_0/2)).
\end{aligned}$$

(d) Expanding $(1-m\sin^2\phi)^{-1/2}$ by the binomial theorem, and noting $\binom{-1/2}{n} = (-1)^n(2n-1)!!/(2n)!!$, we have

$$K(m) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} m^n \int_0^{\pi/2} \sin^{2n}\phi\, d\phi,$$

and the integral is

$$\int_0^{\pi/2} \sin^{2n}\phi\, d\phi = \int_0^{\pi/2} \left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right)^{2n} d\phi = \frac{\pi}{2}\binom{2n}{n} 2^{-2n} = \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!},$$

so

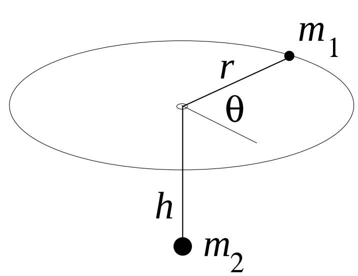
$$K(m) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 m^n.$$

For $\theta_0 = 60°$, $m = \sin^2(\theta_0/2) = 1/4$,

$$\frac{K(m)}{K(0)} = 1 + \left(\frac{1}{2}\right)^2 \frac{1}{4} + \left(\frac{3}{8}\right)^2\left(\frac{1}{4}\right)^2 + \left(\frac{15}{48}\right)^2\left(\frac{1}{4}\right)^3 + \ldots \approx 1.0728$$

3.3 A particle of mass m_1 lies on a frictionless horizontal table with a tiny hole in it. An inextensible massless string attached to m_1 goes through the hole and is connected to another particle of mass m_2, which moves vertically only. Give a full set of generalized unconstrained coordinates and write the Lagrangian in terms of these. Assume the string remains taut at all times and that the motions in question never have either particle reaching the hole, and there is no friction of the string sliding at the hole.
Are there ignorable coordinates? Reduce the problem to a single second order differential equation. Show this is equivalent to single particle motion in one dimension with a potential $V(r)$, and find $V(r)$.

Solution 3.3: The length of the string, $r + h = \ell$ is a constraint, so if we use polar coordinates r and θ for the mass on the table, the remaining coordinate, the height h of the hanging mass, is determined. Thus



$$\begin{aligned} T &= \frac{1}{2}m_1(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}m_2\dot\ell^2 \\ &= \frac{1}{2}(m_1+m_2)\dot r^2 + \frac{1}{2}m_1 r^2\dot\theta^2, \\ U &= K_1 - m_2 g\ell = K + m_2 g r, \end{aligned}$$

where K_1 and K are constants (with $K = K_1 - mg\ell$). Thus Lagrange's equations are

$$\begin{aligned} (m_1+m_2)\ddot r - m_1 r\dot\theta^2 + m_2 g &= 0, \\ \frac{d}{dt}m_1 r^2\dot\theta = 0. \end{aligned}$$

The second equation tells us that

$$L := m_1 r^2 \dot\theta,$$

which is the angular momentum about the vertical through the hole, is conserved. Then we can rewrite

$$\dot\theta = \frac{L}{m_1 r^2}$$

and the first equation becomes

$$(m_1+m_2)\ddot r - \frac{L^2}{m_1} r^{-3} + m_2 g = 0.$$

Here we have an effective force $F = L^2/m_1 r^3 - mg$ and an effective mass $M = m_1 + m_2$, and the problem can be described in terms of an effective potential $V(r) = (L^2/2m_1 r^2) + m_2 g r$, and the total energy $E(r,\dot r) = \frac{1}{2}M\dot r^2 + V(r)$ is a conserved, first integral of the motion.

3.4 Consider some intelligent bugs who live on a turntable which, according to inertial observers, is spinning at angular velocity ω about its center. At any one time, the inertial observer can describe the points on the turntable with polar coordinates r, ϕ. If the bugs measure distances between two objects at rest with respect to them, at infinitesimally close points, they will find

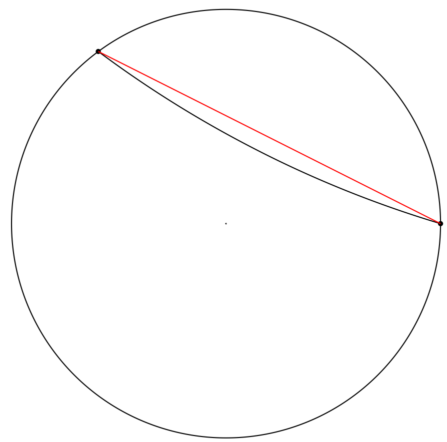
$$d\ell^2 = dr^2 + \frac{r^2}{1-\omega^2 r^2/c^2} d\phi^2,$$

because their metersticks shrink in the tangential direction and it takes more of them to cover the distance we think of as $rd\phi$, though their metersticks agree with ours when measuring radial displacements.

The bugs will declare a curve to be a geodesic, or the shortest path between two points, if $\int d\ell$ is a minimum. Show that this requires that $r(\phi)$ satisfies

$$\frac{dr}{d\phi} = \pm \frac{r}{1-\omega^2 r^2/c^2} \sqrt{\alpha^2 r^2 - 1},$$

where α is a constant.



Straight lines to us and to the bugs, between the same two points.

Solution 3.4: If we consider the path as $r(\phi)$, the length is given by

$$\ell = \int \sqrt{\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}} d\phi,$$

where $\dot r = dr/d\phi$, and $\Omega := \omega/c$. Think of ϕ as time and this the action for a lagrangian

$$L = \sqrt{\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}},$$

which has no explicit time (ϕ) dependence, and therefore has a conserved hamiltonian

$$H = p\dot r - L, \qquad \text{where } p = \frac{\partial L}{\partial \dot r} = \frac{\dot r}{\sqrt{\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}}}.$$

So

$$H = \frac{\dot r^2}{\sqrt{\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}}} - \sqrt{\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}}$$

is a constant. Squaring this gives

$$\begin{aligned} H^2 &= \frac{\dot r^4}{\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}} - 2\dot r^2 + \left(\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}\right) \\ &= \frac{\dot r^4}{\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}} - \dot r^2 + \frac{r^2}{1-\Omega^2 r^2} \end{aligned}$$

$$\begin{aligned} H^2\left(\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}\right) &= \dot r^{\,4} - \dot r^2\left(\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}\right) + \frac{r^2}{1-\Omega^2 r^2}\left(\dot r^2 + \frac{r^2}{1-\Omega^2 r^2}\right) \\ &= \frac{r^4}{(1-\Omega^2 r^2)^2} \end{aligned}$$

With $\alpha^2 = 1/H^2 + \Omega^2$, this gives

$$\dot r = \pm \frac{r}{1-\Omega^2 r^2}\sqrt{\alpha^2 r^2 - 1}.$$
