Physics 507 Fall, 2010

Homework Solution #1

- 1. From the force law $\vec{F}(\vec{r}) = -GM_E m\vec{r}/r^3$, if there is a potential, then
$$U(\vec{r}_1) - U(\vec{r}_2) = \int_{r_1}^{r_2} \vec{F}\cdot d\vec{r}$$
for any path. As every path lying on the surface of a sphere $|r|$ =const. is perpendicular to the radius, and hence the force, at every point of the path, $\vec{F}\cdot d\vec{r} = 0$ along the path and the potentials are the same. Thus $U(\vec{r})$ is the same at every point on a given spherical surface, and $U(\vec{r})$ is a function only of the radius $|r|$. To find that function, consider $\vec{r}_1$ and $\vec{r}_2$ along the positive x-axis, so
$$U(\vec{r}_1) - U(\vec{r}_2) = \int_{x_1}^{x_2} (-GM_e m)\frac{dx}{x^2} = \left.\frac{GM_e m}{x}\right|_{x_1}^{x_2} = \frac{GM_e m}{x_2} - \frac{GM_e m}{x_1}.$$
Choose our arbitrary constant so $\lim_{x_2\to\infty} U(x_2) = 0$, giving
$$U(\vec{r}) = -\frac{GM_e m}{|r|}.$$
To reach infinity, $E = T + U = \frac{1}{2}mv_\infty^2 + U(\infty) \geq U(\infty) = 0$, so at the surface of the Earth the energy is
$$E = \frac{1}{2}mv_E^2 - \frac{GM_e m}{R_E} \geq 0,$$
so $v_E^2 \geq 2GM_E/R_e$, or the velocity $v_E \geq v_e := \sqrt{2GM_E/R_e} = 11,200$ m/s, the escape velocity.

- 2. Consider the system of particles which consists of everything in the rocket at time t, with a total mass $M(t)$, all moving with velocity $\vec{v}(t)$. It is essential to keep in mind that the system considered in our discussion always consists of the same atoms as it evolves in time. So at time $t + \Delta t$, our system consists not only of a rocket with some fuel in it of total mass $M(t+\Delta t)$, moving at velocity $\vec{v}(t+\Delta t)$, but also of a string of fuel of mass $M(t) - M(t+\Delta t) \approx -\Delta t \times dM/dt$, with a velocity $\vec{v}(t) + \vec{u}$. [Note that dM/dt is negative.] If the external force on the system is $\vec{F}(t)$,
$$\begin{aligned}\Delta P &= M(t+\Delta t)\vec{v}(t+\Delta t) - \Delta t\frac{dM}{dt}(\vec{v}(t)+\vec{u}) - M(t)\vec{v}(t)\\ &\approx \Delta t\left(\frac{dM}{dt}\vec{v} + M\frac{d\vec{v}}{dt} - \frac{dM}{dt}[\vec{v}(t)+\vec{u}]\right)\\ &= \vec{F}\Delta t\end{aligned}$$
Thus
$$M\frac{d\vec{v}}{dt} - \frac{dM}{dt}\vec{u} = \vec{F}(t).$$
If $\vec{F}(t)$ is a uniform gravitational field in the $-\hat{e}_z$ direction, $\vec{F}(t) = -Mg\hat{e}_z$, we have $\frac{d\vec{v}}{dt} - \frac{1}{M}\frac{dM}{dt}\vec{u} = -g\hat{e}_z\Delta t$. Assuming all motion is in the z direction, with $\vec{u} = -u\hat{e}_z$, we have,
$$\frac{dv}{dt} = -u\frac{1}{M}\frac{dM}{dt} - g,$$
which can be integrated to give
$$v(t) - v(0) = u\ln[M(0)/M(t)] - gt.$$
To reach a high velocity, one needs to burn up most of the mass. Even without gravity retarding the acceleration, to reach $v = 11,200$ m/s, with $u = 2000$ m/s, would require $M(0)/M(t) = 270$. This assumes the burning is instantaneous. If the burning takes longer, the $-gt$ term just makes the situation worse. [As a limiting case, it might burn the mass just quickly enough to balence the force of gravity, in which case v never changes.]