These things were listed in More, on Oct 30, 1997:
`chaplagr/dissipation.tex` has nothing in it except a header.
`chaplagr/varconstr.tex` Variations with constraints we didn't reach a satisfactory conclusion:

### 0.0.1 Variations with constraints

In our discussion of Hamilton's Principle, we assumed that we could work with a set of generalized coordinates $q_i$ which were unconstrained. This set may have been what was left after eliminating some holonomic constraints from the original coordinates, as for example $\vec{r} \Rightarrow \{q\} = (\theta, \phi)$ for the gimballed rod, but in any case it was essential to extracting (**??**) from $\delta I = 0$ that the $\delta q_i$ are independent arbitrary functions of time. In considering a disk rolling on a plane (section **??**) we saw that there are sometimes anholonomic constraints of the form

$$\Phi_\alpha(q_j, \dot{q}_\ell, t) = 0, \alpha = 1...k, \tag{1}$$

which cannot be solved to eliminate some of the $q_j$ and leave only independent coordinates. How can we make a variation which respects such constraints?

First let us consider the simpler situation in which the variation is of a finite number of coordinates and not a functional variation. Suppose we want to maximize $f(x_1, ...x_N)$ subject to constraints $\phi_\alpha(x_1, ...x_N) = 0, \alpha = 1...k, k < N$. Generically, at an arbitrary point $\vec{x}$, the $k$ vectors $\vec{\nabla}\phi_\alpha$ will be linearly independent and will span a $k$-dimensional subspace $V_U$. The variations[1] allowed by the constraints will be perpendicular to this subspace, and form a $N - k$ dimensional subspace $V_A$. The condition that $f$ is a stationary point under variations on $V_A$ says that $\vec{\nabla} f = \sum_\alpha \lambda_\alpha \vec{\nabla}\Phi_\alpha$ for some set of constants $\lambda_\alpha$, for arbitrary variations of $\vec{x}$. We may thus solve the $N + k$ equations

$$\begin{aligned} \vec{\nabla}\left(f - \sum_\alpha \lambda_\alpha \Phi_\alpha\right) &= 0, \\ \phi_\alpha &= 0, \end{aligned}$$

[1] The variations considered here are infinitesimal, and the subspaces discussed here are, strictly speaking, statements about the tangent space, which is a vector space on which $\delta f$ is a linear function of $\delta\vec{x}$.

for the $N+k$ variables $\vec{x}$ and $\lambda_\alpha$. If there are several stationary points, the $\lambda_\alpha$ need not be the same, so in general they will depend on the solution found. They are called **Lagrange multipliers**.

For variation of the action subject to the constraints (1), we might try to do the corresponding thing, which is to ask that the action $\tilde{S} = \int \left(L - \sum_\alpha \lambda_\alpha(t)\Phi_\alpha\right) dt$ be stationary under all variations $\delta q(t)$, subject to the additional constraints $\Phi_\alpha = 0$.

**This appears not to work.!!!**

The first requirement leads to the $N$ conditions

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - \sum_\alpha \frac{d}{dt}\left(\lambda_\alpha \frac{\partial \Phi_\alpha}{\partial \dot{q}_i}\right) + \sum_\alpha \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial q_i} = 0. \tag{2}$$

If the constraints $\Phi_\alpha$ are linear in the velocities,

$$\Phi_\alpha(\dot{q}_i, q_i, t) = \sum_j a_{\alpha j}(q,t)\dot{q}_j + b_\alpha(q,t),$$

then

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_\alpha \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial \dot{q}_i}\right) - \sum_\alpha \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial q_i} \\
&= \frac{d}{dt}\left(\sum_\alpha \lambda_\alpha(t) a_{\alpha i}(q,t)\right) - \sum_{\alpha j} \lambda_\alpha \frac{\partial a_{\alpha j}}{\partial q_i}\dot{q}_j - \sum_{\alpha j} \lambda_\alpha \frac{\partial b_\alpha(q,t)}{\partial q_i} \\
&= \sum_\alpha \frac{d\lambda_\alpha(t)}{dt} a_{\alpha i}(q,t) + \sum_\alpha \lambda_\alpha(t)\left(\dot{q}_j \frac{\partial a_{\alpha i}}{\partial q_j} + \frac{\partial a_{\alpha i}}{\partial t}\right) \\
&\qquad - \sum_{\alpha j} \lambda_\alpha \frac{\partial a_{\alpha j}}{\partial q_i}\dot{q}_j - \sum_{\alpha j} \lambda_\alpha \frac{\partial b_\alpha(q,t)}{\partial q_i}
\end{aligned}$$

...

`chaplagr/wolinsky` a note about pulling on a string on a ball on a string:

`\input wolinsky`

`chaplagr/nosubst,tex` dont plug eq of motion into Lagrange

When we have a Lagrangian with an ignorable coordinate, say $\theta$, and therefore a conjugate momentum $P_\theta$ which is conserved and can be considered a constant, we are able to reduce the problem to one involving one fewer degrees of freedom. That is, one can substitute into the other differential equations the value of $\dot{\theta}$ in terms of $P_\theta$ and other degrees of freedom, so that $\theta$ and its derivatives no longer appear in the equations of motion. For example, consider the two dimensional isotropic harmonic oscillator,

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)-\frac{1}{2}k\left(x^2+y^2\right) \\ &= \frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-\frac{1}{2}kr^2 \end{aligned}$$

in polar coordinates. The equations of motion are

$$\begin{aligned} \dot{P}_\theta &= 0, \quad \text{where } P_\theta = mr^2\dot{\theta}, \\ m\ddot{r} &= -kr+mr\dot{\theta}^2 \Longrightarrow m\ddot{r} = -kr + P_\theta^2 \Big/ m^2r^3. \end{aligned}$$

The last equation is now a problem in the one degree of freedom $r$.

One might be tempted to substitute for $\dot{\theta}$ into the Lagrangian and then have a Lagrangian involving one fewer degrees of freedom. In our example, we would get

$$L = \frac{1}{2}m\dot{r}^2+\frac{P_\theta^2}{2m^2r^2}-\frac{1}{2}kr^2,$$

which gives the equation of motion

$$m\ddot{r} = -\frac{P_\theta^2}{m^2r^3}-kr.$$

**This is wrong**

Notice that the last equation has the sign of the $P_\theta^2$ term reversed from the correct equation. Why did we get the wrong answer? In deriving the Lagrange equation which comes from varying $r$, we need

$$\frac{d}{dt}\left.\frac{\partial L}{\partial \dot{r}}\right|_{r,\theta,\dot{\theta}} = \left.\frac{\partial L}{\partial r}\right|_{\dot{r},\theta,\dot{\theta}}.$$

But we treated $P_\theta$ as fixed, which means that when we vary $r$ on the right hand side, we are not holding $\dot{\theta}$ fixed, as we should be. While we often write partial derivatives without specifying explicitly what is being held fixed, they are not defined with such a specification, which we are expected to understand implicitly. However, there are several examples in Physics, such as thermodynamics, where this implicit understanding can be unclear, and the results may not be what was intended.

`chappert/adiabnot.tex` notes triggered by Harry, not yet integrated (really?) No, I think it is all in adiab.tex

Here are files that appear in the exers subdirectory but not in the input exercises.tex file:
chappk, chaplagr: None
chap2b/exers/hodograph.tex:

**0.1** For the Kepler problem we have the relative position tracing out an ellipse. What is the curve traced out by the momentum in momentum space? Show that it is a circle centered at $\vec{l} \times \vec{A}/L^2$, where $\vec{L}$ and $\vec{A}$ are the angular momentum and Runge-Lenz vectors respectively.

**ex:hodograph** From $\vec{A} = \vec{p} \times \vec{L} - \mu K \vec{r}/r$, we have

$$\begin{aligned}
\vec{L} \times \vec{A} &= \vec{L} \times (\vec{p} \times \vec{L}) - \frac{\mu^2 K}{r} (\vec{r} \times \vec{p}) \times \vec{r} \\
&= L^2 \vec{p} - \vec{L}(\vec{p} \cdot \vec{L}) - \frac{\mu^2 K}{r} (r^2 \vec{p} - \vec{r}(\vec{r} \cdot \vec{p})) \\
&= L^2 \vec{p} - \frac{\mu^2 K}{r} (r^2 \vec{p} - \vec{r}(\vec{r} \cdot \vec{p}))
\end{aligned}$$

as $\vec{p}$ is perpendicular to $\vec{L}$. So

$$p - \frac{\vec{L} \times \vec{A}}{L^2} = \frac{\mu^2 K}{r L^2} (r^2 \vec{p} - \vec{r}(\vec{r} \cdot \vec{p}))$$

and

$$\left( p - \frac{\vec{L} \times \vec{A}}{L^2} \right)^2 = \frac{\mu^4 K^2}{r^2 L^4} (r^4 p^2 - r^2 (\vec{r} \cdot \vec{p})^2) = \frac{\mu^4 K^2}{L^4} (\vec{r} \times \vec{p})^2 = \frac{\mu^2 K^2}{L^2},$$

which is a constant. Of course $\vec{p}$ is confined to the plane perpendicular to $\vec{L}$, so the path is a circle of radius $\mu K/|L|$ centered at $\vec{L} \times \vec{A}/L^2$.
chaprigid/exers/asymtop.tex:

**0.2** Write the Lagrangian for the asymmetric top, with $I_1 \neq I_2$. How many constants of the motion can you find for this problem?

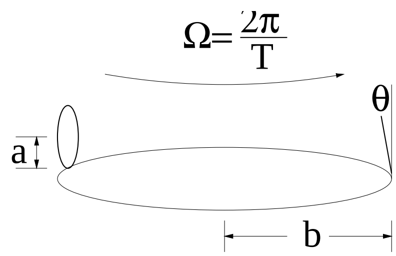
**ex:topelliptic** In the body fixed principal axis coordinates,

$$\begin{aligned}
L &= \frac{1}{2}\sum_i I_i\omega_i^2 - Mg\ell\cos\theta \\
&= \frac{1}{2}I_1(\dot\theta\sin\psi - \dot\phi\sin\theta\cos\psi)^2 + \frac{1}{2}I_2(\dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi)^2 \\
&\qquad +\frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - Mg\ell\cos\theta \\
&= \frac{1}{2}I_1(\dot\theta^2\sin^2\psi + \dot\phi^2\sin^2\theta\cos^2\psi - 2\dot\theta\dot\phi\sin\theta\sin\psi\cos\psi) \\
&\quad +\frac{1}{2}I_2(\dot\theta^2\cos^2\psi + \dot\phi^2\sin^2\theta\sin^2\psi + 2\dot\theta\dot\phi\sin\theta\sin\psi\cos\psi) \\
&\qquad +\frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - Mg\ell\cos\theta \\
&= \frac{1}{2}\dot\theta^2(I_1\sin^2\psi + I_2\cos^2\psi) + \frac{1}{2}\dot\phi^2\sin^2\theta(I_1\cos^2\psi + I_2\sin^2\psi) \\
&\quad -\dot\theta\dot\phi(I_1 - I_2)\sin\theta\sin\psi\cos\psi \\
&\qquad +\frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - Mg\ell\cos\theta \\
&= \frac{1}{4}(I_1 + I_2)(\dot\theta^2 + \dot\phi^2\sin^2\theta) \\
&\quad +\frac{1}{4}(I_1 - I_2)(-\dot\theta^2\cos 2\psi + \dot\phi^2\sin^2\theta\cos 2\psi - 2\dot\theta\dot\phi\sin\theta\sin 2\psi) \\
&\qquad +\frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - Mg\ell\cos\theta
\end{aligned}$$

This simplification hasn't helped much. What are the constants of the motion? As the Lagrangian has no explicit time or $\phi$ dependence, the energy and $p_\phi$ are conserved. In the symmetric case where $I_1 = I_2$, there was also no explicit dependence on $\psi$, but that is not true here.

chaprigid/exers/rollpenny.tex

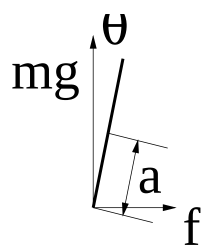
**0.3** A thin disk of uniform material and radius $a$, rolls around a circle of radius $b$ on a horizontal surface. It is tilted from the vertical by exactly the angle $\theta$ required so that it continues in this situation, make circuits around the circle of radius $b$ in time $T$ Assuming that $\theta \ll 1$ and that the frictional forces do no work, find the angle $\theta$ in terms of the other parameters.




**ex:rollpenny** The disk is rotating around the center of the large circle with angular velocity $\Omega = 2\pi/T$ at the same time it is rotating about its center of mass with an angular velocity $\omega$. As the point of contact with the ground is not moving, and as the smallness of $\theta$ permits us to take the distance the center of mass is from the axis to be $b$, we have $a\omega = b\Omega$. The angular momentum has a piece coming from the rotation of the center of mass about the large circle, a constant in the upwards direction, and a contribution from the rotation of the disk about its center of mass, $I\omega = \frac{1}{2}a^2m\omega$, which has a horizontal component towards the center of the circle of magnitude $L_r = \frac{1}{2}a^2m\omega\cos\theta \sim \frac{1}{2}a^2m\omega$. As this vector is rotating with angular velocity $\Omega\hat{e}_z$, $\frac{d\vec{L}}{dt} = \frac{1}{2}a^2m\omega\Omega = \frac{1}{2}abm\Omega^2$ in the direction the disk is rolling.

This change in $\vec{L}$ must be caused by the net torque about the center of mass. $\dot{\vec{L}} = af\cos\theta - mg\sin\theta \sim af - mg\theta$, where $f$ is the centripetal force required, $f = m\Omega^2 b$, so $abm\Omega^2 - mg\theta = \frac{1}{2}abm\Omega^2$, or

$$\theta = \frac{3}{2}ab\Omega^2.$$




chapso: none
chapham/exers/canona.tex:

**0.4** Consider the unusual Hamiltonian for a one-dimensional problem

$$H = \omega(x^2 + 1)p,$$

where $\omega$ is a constant.

**(a)** Find the equations of motion, and solve for $x(t)$.

**(b)** Consider the transformation to new phase-space variables $P = \alpha p^{\frac{1}{2}}$, $Q = \beta x p^{\frac{1}{2}}$. Find the conditions necessary for this to be a canonical transformation, and find a generating function $F(x, Q)$ for this transformation.

**(c)** What is the Hamiltonian in the new coordinates?

**ex:canona**

**(a)** $\dot{x} = \dfrac{\partial H}{\partial p} = \omega(x^2+1)$, $\dot{p} = -\dfrac{\partial H}{\partial x} = -2\omega x p$. From the first equation,

$$\int \frac{dx}{x^2+1} = \int \omega dt, \qquad \text{or} \quad \tan^{-1} x = \omega t + \delta, \quad \Rightarrow \quad x = \tan(\omega t + \delta).$$

**(b)** The new variables $Q = \beta x p^{\frac{1}{2}}$ and $P = \alpha p^{\frac{1}{2}}$ are canonical if

$$[Q, P] = 1 = \beta p^{\frac{1}{2}} \frac{\alpha}{2p^{\frac{1}{2}}} = \frac{\alpha\beta}{2},$$

so all that is needed is $\beta = 2/\alpha$. For a generating function of type 1, we need to solve $p = Q^2\beta^{-2}x^{-2} = \alpha^2 Q^2/4x^2$, but

$$p = \left.\frac{\partial F}{\partial x}\right|_Q \Rightarrow F(x, Q) = -\frac{\alpha^2 Q^2}{4x} + f(Q).$$

Then $P = -\partial F/\partial Q = \frac{\alpha^2 Q}{2x} + f'(Q) = \alpha p^{\frac{1}{2}} + f'(Q) \Rightarrow f'(Q) = 0$, and we can drop the unknown constant $f$.

**(c)** As the transformation is not time-dependent, the Hamiltonian is obtained simply by substituting $p = P^2/\alpha^2$ and $x = Q/(\beta p^{\frac{1}{2}}) = \alpha Q/(\beta P) = \frac{1}{2}\alpha^2 Q/P$. Thus we have

$$H = \omega \left( \frac{\alpha^2 4 Q^2}{4P^2} + 1 \right) \frac{P^2}{\alpha^2} = \omega \left( \frac{\alpha^2 Q^2}{4} + \frac{P^2}{\alpha^2} \right).$$

If we choose $\omega = \sqrt{k/m}$ and $\alpha = (4km)^{1/4}$, this becomes

$$H = \frac{k}{2} Q^2 + \frac{1}{2m} P^2,$$

our standard harmonic oscillator. Note if $Q = A\sin(\omega t + \delta)$, $P = Am\dot{Q} = \frac{1}{2}\alpha^2 A\cos(\omega t + \delta)$, and $x = \frac{1}{2}\alpha^2 Q/P = \tan(\omega t + \delta)$ in agreement with our previous solution.

chappert/exers/adiabpend.tex:

**0.5** Consider a mass $m$ hanging at the end of a length of string which passes through a tiny hole, forming a pendulum. The length of string below the hole, $\ell(t)$ is slowly shortened by someone above the hole pulling on the string. How does the amplitude (assumed small) of the oscillation of the pendulum depend on time? (Assume there is no friction).

**ex:adiabpend** The angular frequency of oscillation is $\omega(t) = \sqrt{g/\ell(t)}$, and the displacement is $q(t) = A(t)\sin[\omega(t)t + \delta(t)]$, where $A$, $\ell$ and $\delta$ all vary slowly during one oscillation. The momentum is $p(t) = m\dot{q}(t) \approx mA(t)\omega(t)\cos[\omega(t)t + \delta(t)]$. The action is

$$J(t) = \int p dq = A^2(t)\omega^2(t)\int_0^{2\pi/\omega} \cos^2(\omega t + \delta)dt = \pi A^2(t)\omega(t),$$

where we have extracted the slowly varying quantities from the short time integral. But if the length of the string is varied slowly,

$$\frac{\dot{\ell}}{\ell} \ll \frac{1}{\tau} = \frac{1}{2\pi}\omega(t) = \frac{1}{2\pi}\sqrt{g/\ell(t)}, \qquad \text{or} \quad \dot{\ell} \ll \sqrt{g\ell},$$

the action is an adiabatic invariant, so $A^2(t)\omega(t)$ is constant, or

$$A(t) \propto [\omega(t)]^{-1/2} \propto [\ell(t)]^{1/4}.$$

chappert/exers/defines.tex:

**0.6** Define precisely or explain clearly what is meant by
a) apsidal angle,
b) closed 2-form,
c) seperatrix,
d) anholonomic constraint,
e) invariant set of states,
f) natural symplectic structure on phase space.

**ex:pertdefs** (a) The apsidal angle, for bound state motion in the two body central force problem, is the angle subtended at the center of force, between any

perigee and the following apogee, and also the angle between any apogee and the following perigee.
b) A closed 2-form $\omega = \sum_{i<j} B_{ij} dx_i \wedge dx_j$ is a 2-form on which the exterior derivative vanishes,

$$d\omega = \sum_k \sum_{i<j} B_{ij,k} dx_k \wedge dx_i \wedge dx_j = 0.$$

This implies (for $B_{ij} = -B_{ji}$) that $B_{ij,k} + B_{ki,j} + B_{jk,i} = 0$.
c) A seperatrix in a 2nd order dynamical system with a conserved quantity, is an invariant set of states, terminating at an unstable fixed point, which divides phase space into regions with qualitatively different behaviors. For example, in the figure on P. 36 of my book, the egg shape line which reaches a cusp at the fixed point is one, and each of the two lines which terminate at the fixed point are two others.
d) An anholonomic constraint on the coordinates and velocities of a physical system is a constraint which can not be rewritten in terms of an equation involving only the coordinates and some arbitrary fixed parameters. It might be an equation involving the velocities or it might be an inequality.
e) An invariant set of states in phase space is a minimum set of points in phase space which are mapped into each other under the dynamical motion, forwards or backwards through any time interval.
f) The natural symplectic structure on phase space is the 2-form $\omega_2 = \sum_i dp_i \wedge dq_i$. Despite its representation in terms of a particular set of coordinates and conjugate momenta, it is invariant under any canonical transformation.

chappert/exers/fsol.tex:

**0.7** *what is the question?*

**ex:LandA** By definition, the momentum fields conjugate to $\vec{A}$ and $\phi$ are given by

$$\begin{aligned}\pi_i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \frac{\partial \mathcal{L}}{\partial E_i} \frac{\partial E_i}{\partial \dot{A}_i} = -\frac{1}{c} E_i \\ \pi_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0.\end{aligned}$$

The last condition is quite unusual, because will it is not unusual to get an equation of motion which says the momentum is constant, getting one which sets it to a specific value means the the momentum cannot be chosen arbitrarily for an initial condition, or equivalently as a point in phase space. This peculiarity has to do with the gauge invariance in electromagnetism, but we can't go into that in more detail here, except as mentioned below.

The Lagrange equations are generally

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\eta_i} + \sum_j \frac{d}{dx_j}\frac{\partial \mathcal{L}}{\partial \partial\eta_i/\partial x_j} - \frac{\partial \mathcal{L}}{\partial \eta_i} = 0.$$

In this case $\eta_i = (\phi, A_i)$, and in each case the Lagrange density depends only on the derivatives of the $\eta_i$ and not on $\eta_i$ itself. We need

$$\frac{\partial \mathcal{L}}{\partial\partial\phi/\partial x_j} = \sum_i \frac{\partial \mathcal{L}}{\partial E_i}\frac{\partial E_i}{\partial\partial\phi/\partial x_j} = -E_j,$$

$$\frac{\partial \mathcal{L}}{\partial\partial A_i/\partial x_j} = \sum_k \frac{\partial \mathcal{L}}{\partial B_k}\frac{\partial B_k}{\partial\partial A_i/\partial x_j} = (-B_k)(\epsilon_{kij}).$$

For $\phi$ this gives

$$\frac{d}{dt}0 - \sum_j \frac{d}{dx_j}E_j = 0, \text{ or } \vec\nabla\cdot E = 0,$$

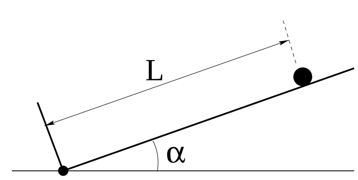
which is Gauss' law in empty space, while for $A_i$ we have

$$0 = \frac{d}{dt}\left(-\frac{1}{c}E_i\right) + \sum_j \frac{\partial}{\partial x_j}\left(\epsilon_{ijk}B_k\right) = \left(\vec\nabla \times B - \frac{1}{c}\frac{d\vec E}{dt}\right)_i.$$

These, of course, are two of Maxwell's equations, the ones that come from setting $J_\mu = 0$, for empty space. The other two come simply from the definition of $\vec E$ and $\vec B$ in terms of $\vec A$ and $\phi$.

chappert/exers/ramp.tex:

**0.8** A particle of mass $m$ slides without friction on a flat ramp which is hinged at one end, at which there is a fixed wall. When the mass hits the wall it is reflected perfectly elastically. An external agent changes the angle $\alpha$ very slowly compared to the interval between successive times at which the particle reaches a maximum height. If the angle varies from from an initial value of $\alpha_I$ to a final value $\alpha_F$, and if the maximum excursion is $L_I$ at the beginning, what is the final maximum excursion $L_F$?



**ex:adiabramp** For any fixed angle $\alpha$ the Hamiltonian is

$$H = \frac{p^2}{2m} + mgx \sin\alpha,$$

where $x$ is the distance from the fixed wall. In the course of one bounce the energy is approximately conserved,

$$E = \frac{1}{2}mv^2 + mgx\sin\alpha = mgL\sin\alpha,$$

so

$$\begin{aligned} v &= \sqrt{2g\sin\alpha(L-x)} \\ J &= \int p\,dx = m\int v(x)dx = m\sqrt{2g\sin\alpha} \times 2\int_0^L \sqrt{L-x}dx \\ &= \frac{4m}{3}\sqrt{2gL^3\sin\alpha} \end{aligned}$$

The action is an adiabatic invariant, so $J$ is the same at the start and the end,

$$L_I^3 \sin\alpha_I = L_F^3 \sin\alpha_F, \quad \text{or} \quad L_F = L_I\left(\frac{\sin\alpha_I}{\sin\alpha_F}\right)^{1/3}.$$

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