0.1 Consider the harmonic oscillator $H = p^2/2m + \frac{1}{2}m\omega^2q^2$ as a perturbation on a free particle $H_0 = p^2/2m$. Find Hamilton's Principle Function $S(q,P)$ which generates the transformation of the unperturbed hamiltonian to Q, P the initial position and momentum. From this, find the Hamiltonian $K(Q,P,t)$ for the full harmonic oscillator, and thus equations of motion for Q and P. Solve these iteratively, assuming $P(0) = 0$, through fourth order in ω. Express q and p to this order, and compare to the exact solution for an harmonic oscillator.

ex:timedepp For a free particle $p = p(0) = P$, $q(t) = q(0) + p(0)t/m = Q + Pt/m$, so $q = Q + Pt/m$, which is generated by $S(q,P,t) = qP - P^2t/2m$. The full hamiltonian in these coordinates is

$$K(Q,P,t) = H(q,p) + \frac{\partial S}{\partial t} = \frac{m}{2}\omega^2\left(Q + \frac{Pt}{m}\right)^2,$$

giving the equations of motion

$$\dot{Q} = \omega^2 t\left(Q + \frac{Pt}{m}\right), \qquad \dot{P} = -m\omega^2\left(Q + \frac{Pt}{m}\right).$$

Solving iteratively $\zeta_{n+1}(t) = \zeta(0) + \int_0^t \dot{\zeta}_n(t')dt'$, with $P(0) = 0$, we get

$$\begin{array}{cc} Q_1 = \left(1 + \frac{1}{2}\omega^2t^2\right)Q(0) & P_1 = -m\omega^2 tQ(0) \\ Q_2 = \left(1 + \frac{1}{2}\omega^2t^2 - \omega^4t^4/8\right)Q(0) & P_2 = \left(-m\omega^2t + m\omega^4t^3/6\right)Q(0) \end{array}.$$

The new and old momenta are the same, but the coordinate q now has an expansion

$$q_2(t) = Q_2 + P_2t/m = \left(1 - \frac{1}{2}\omega^2t^2 + \frac{\omega^4t^4}{24}\right)Q(0),$$

which is just the first few terms of the power series expansion of the exact solution $q(t) = Q(0)\cos\omega t$. Similarly the expansion for $p(t)$ is the beginning of the expansion for the exact solution $p(t) = -m\omega Q(0)\sin\omega t$.

0.2 Consider the Kepler problem in two dimensions. That is, a particle of (reduced) mass μ moves in two dimensions under the influence of a potential

$$U(x,y) = -\frac{K}{\sqrt{x^2+y^2}}.$$

This is an integrable system, with two integrals of the motion which are in involution. In answering this problem you are expected to make use of the explicit solutions we found for the Kepler problem.
a) What are the two integrals of the motion, F_1 and F_2, in more familiar terms and in terms of explicit functions on phase space.
b) Show that F_1 and F_2 are in involution.
c) Pick an appropriate $\eta_0 \in \mathcal{M}_{\vec{f}}$, and explain how the coordinates $\vec{t}$ are related to the phase space coordinates $\eta = g^{\vec{t}}(\eta_0)$. This discussion may be somewhat qualitative, assuming we both know the explicit solutions of Chapter 3, but it should be clearly stated.
d) Find the vectors $\vec{e}_i$ which describe the unit cell, and give the relation between the angle variables ϕ_i and the usual coordinates η. One of these should be explicit, while the other may be described qualitatively.
e) Comment on whether there are relations among the frequencies and whether this is a degenerate system.

ex:intsyskep If we use polar coordinates, the potential energy is $U = -K/r$, the kinetic energy is $T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2$. The momenta conjugate to r and θ are then $p_r = \mu\dot{r}$ and $p_\theta = \mu r^2\dot{\theta}$ respectively, and as the Lagrangian has no explicit time dependence and θ is an ignorable coordinate, the Hamiltonian $H = p_r^2/2m + p_\theta^2/2mr^2 + K/r$ and p_θ are conserved, *i.e.* are integrals of the motion. Note p_θ is the angular momentum about the z-axis, *i.e.* perpendicular to the plane.
a) Thus $F_1 = H$ and $F_2 = p_\theta$ are two integrals of the motion.
b) As θ does not appear in H, $[H, p_\theta] = 0$, and the two are in involution.
c) For a given $\vec{f} = (E, L)$, the manifold $\mathcal{M}_{\vec{f}}$ consists of all points in phase space consistent with that E and $p_\theta = L$. As the semi-major axis and ellipticity of the elliptical orbit is determined by E and L, $\mathcal{M}_{\vec{f}}$ has all points possible for all such ellipses. Thus its projection onto coordinate space is an annulus with perigee and apogee r_p and r_a as the radii. The momentum p_r is given, modulo sign, by $\sqrt{2m(E - U(r))}$, so our invariant torus really is a torus. The remaining momentum, p_θ, is a constant. We may choose η_0 any point on it, so let $\eta_0 = (r_a, 0, 0)$

The other points in $\eta \in \mathcal{M}_{\vec{f}}$ are generated by the canonical transformation $g^{\vec{t}}$, *i.e.* $\eta = g^{\vec{t}}(\eta_0)$. The parameters t_1, t_2 give the parameters by which the generators $F_1 = H$ and $F_2 = p_\theta$ have been applied. The momentum as a generator generates a translation in the conjugate coordinate, so

$\eta = (r, \theta, p_r) = g^{0,t_2}(\eta_0) = (r_a, t_2, 0)$, while the generator H moves the phase space point forwards in time according to the standard newtonian laws. Thus if we solved the Kepler problem for $\vec{r}(0) = r_a \hat{e}_x$, $p_r(0) = 0$, $p_\theta = L$, the solution $(r(t), \theta(t), p_r(t))$ is the value $\eta = g^{t,0}(\eta_0)$.
d) Clearly $\theta \to \theta + 2\pi$ brings us back to the same point in phase space, so there is a periodicity under $t_2 \to t_2 + 2\pi$. We also know the dynamical motion is periodic with period $T = \pi K \sqrt{\mu/2}(-E)^{-3/2}$, so this is the period of t_1. Thus the $\vec{e}_i$ are $(0, 2\pi)$ and $(T, 0)$ respectively, A is diagonal with elements $T/2\pi$ and 1, and the frequencies $\omega_i = (A^{-1})_{i1} = (T/2\pi, 0)$.
e) The relation $\omega_2 = 0$ is a relation among the frequencies, is there independent of the values $\vec{f}$ of the integrals of the motion, so we have here a degenerate system.

0.3 Consider a mass m hanging at the end of a length of string which passes through a tiny hole, forming a pendulum. The length of string below the hole, $\ell(t)$ is slowly shortened by someone above the hole pulling on the string. How does the amplitude (assumed small) of the oscillation of the pendulum depend on time? (Assume there is no friction).

ex:adiabpend The angular frequency of oscillation is $\omega(t) = \sqrt{g/\ell(t)}$, and the displacement is $q(t) = A(t)\sin[\omega(t)t + \delta(t)]$, where A, ℓ and δ all vary slowly during one oscillation. The momentum is $p(t) = m\dot{q}(t) \approx mA(t)\omega(t)\cos[\omega(t)t + \delta(t)]$. The action is

$$J(t) = \int p dq = A^2(t)\omega^2(t) \int_0^{2\pi/\omega} \cos^2(\omega t + \delta) dt = \pi A^2(t)\omega(t),$$

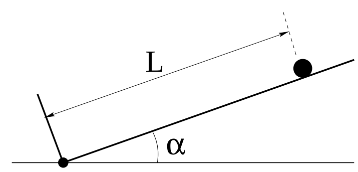
where we have extracted the slowly varying quantities from the short time integral. But if the length of the string is varied slowly,

$$\frac{\dot{\ell}}{\ell} \ll \frac{1}{\tau} = \frac{1}{2\pi}\omega(t) = \frac{1}{2\pi}\sqrt{g/\ell(t)}, \qquad \text{or} \quad \dot{\ell} \ll \sqrt{g\ell},$$

the action is an adiabatic invariant, so $A^2(t)\omega(t)$ is constant, or

$$A(t) \propto [\omega(t)]^{-1/2} \propto [\ell(t)]^{1/4}.$$

0.4 A particle of mass m slides without friction on a flat ramp which is hinged at one end, at which there is a fixed wall. When the mass hits the wall it is reflected perfectly elastically. An external agent changes the angle α very slowly compared to the interval between successive times at which the particle reaches a maximum height. If the angle varies from from an initial value of α_I to a final value α_F, and if the maximum excursion is L_I at the beginning, what is the final maximum excursion L_F?



ex:adiabramp For any fixed angle α the Hamiltonian is

$$H = \frac{p^2}{2m} + mgx\sin\alpha,$$

where x is the distance from the fixed wall. In the course of one bounce the energy is approximately conserved,

$$E = \frac{1}{2}mv^2 + mgx\sin\alpha = mgL\sin\alpha,$$

so

$$\begin{aligned} v &= \sqrt{2g\sin\alpha(L-x)} \\ J &= \int p\,dx = m\int v(x)dx = m\sqrt{2g\sin\alpha}\times 2\int_0^L \sqrt{L-x}dx \\ &= \frac{4m}{3}\sqrt{2gL^3\sin\alpha} \end{aligned}$$

The action is an adiabatic invariant, so J is the same at the start and the end,

$$L_I^3\sin\alpha_I = L_F^3\sin\alpha_F, \quad \text{or} \quad L_F = L_I\left(\frac{\sin\alpha_I}{\sin\alpha_F}\right)^{1/3}.$$

0.5 Consider a particle of mass m and charge q in the field of a fixed electric dipole with moment $\vec{p}$. Using spherical coordinates with the axis in the $\vec{p}$ direction, the potential energy is given by

$$U(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta.$$

There is no explicit t or ϕ dependence, so H and $p_\phi = L_z$ are conserved.
a) Show that

$$A = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + \frac{qpm}{2\pi\epsilon_0}\cos\theta$$

is also conserved.
b) Given these three conserved quantities, what else must you show to find if this is an integrable system? Is it true? What, if any, conditions are there for the motion to be confined to an invariant torus?

ex:dipoleIS a) The Hamiltonian is

$$\begin{aligned} H &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + \frac{1}{4\pi\epsilon_0}\frac{qp}{r^2}\cos\theta \\ &= \frac{p_r^2}{2m} + \frac{1}{2mr^2}A, \end{aligned}$$

where

$$A = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + \frac{qpm}{2\pi\epsilon_0}\cos\theta.$$

A contains neither r nor p_r, so it has zero Poisson bracket with $p_r^2/2m$ and with $1/(2mr^2)$, and with itself, of course, so $[A, H] = 0$. A is also time independent, so

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t} = 0,$$

and A is conserved.
b) The three conserved quantities must be in involution; that is, they must have zero Poisson brackets. p_ϕ and A have already been shown to have zero Poisson brackets with H, but it is also true that, because ϕ does not appear in A, $[A, p_\phi] = 0$, so the three are in involution, and we have an integrable system.

Thus the motion is confined to a region with fixed E, A, and p_ϕ. Then

$$p_r^2 = 2mE - A/(2mr^2),$$

and the allowed region of r is determined by the signs and relative magnitudes of E and A. This rules out $A > 0, E < 0$, as there would then be no possible values of r. If $A < 0, E > 0$, there is a lower bound on $|p_r| = m|\dot{r}|$, so the motion in r cannot be bounded. This is terminating motion at $r = 0$, not an invariant torus. If E and A have the same sign the right hand side vanishes at some r_0, and we have $\dot{r} \propto \sqrt{r_0^{-2} - r^{-2}}$, or

$$t \propto \int \frac{dr}{\sqrt{r_0^{-2} - r^{-2}}} = \int \frac{rdr}{\sqrt{r^2/r_0^2 - 1}} \propto \int \frac{du}{\sqrt{u-1}} = \ln(u-1),$$

which diverges as $u \to 1$, or $r \to r_0$. Thus the motion begins or ends (depending on the sign taken for the square root) at r_0, and ends or begins at either $r = 0$ or $r = \infty$, depending on the signs of E and A. In no case is it an invariant torus, and in no case a bounded region of phase space. Of course, the one point $E = A = 0$, $r =$ constant is a lower dimensional torus.