Physics 504, Lecture 9
Feb. 22, 2010

1 Multipole Expansion, Vector Spherical Harmonics

Last time we derived the expansion of the Green's function for the scalar helmholtz equation, $(\nabla^2 + k^2)\Psi = -\delta(\vec{x} - \vec{x}\,')$, but we observed that it was awkward to use it for each of the *Cartesian* components of the vector potential $\vec{A}$. Indeed, we found for $\ell = 0$ that $\vec{A}$ is dominated by the electric dipole moment, which looks much like an $\ell = 1$ effect. When we discussed the resonant cavity formed by the Earth and its ionosphere, we considered $\Psi = \vec{r} \cdot \vec{E}$ or $\Psi = \vec{r} \cdot \vec{H}$, which is more compatible with using spherical coordinates. Because away from sources $\vec{E}$ and $\vec{H}$ are *both* divergenceless, each of these Ψ's obeys the free Helmholtz equation away from the sources, and can be expanded as we did in lecture 5: Either J9.112

J9.117 $f \to g$

$$\vec{r} \cdot \vec{H}^{(M)}_{\ell m} = \frac{\ell(\ell+1)}{k} g_\ell(kr) Y_{\ell m}(\theta, \phi), \qquad \vec{r} \cdot \vec{E}^{(M)} = 0 \tag{1}$$

$$\text{or} \quad \vec{r} \cdot \vec{E}^{(E)}_{\ell m} = -Z_0 \frac{\ell(\ell+1)}{k} g_\ell(kr) Y_{\ell m}(\theta, \phi), \qquad \vec{r} \cdot \vec{H}^{(E)} = 0. \tag{2}$$

for magnetic multipole modes (M) (Eq. 1) or electric multipole modes (E) (Eq. 2). In either case g_ℓ satisfies the spherical Bessel equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} + k^2 \right) g_\ell(kr) = 0$$

with solutions outside the source region proportional to $h^{(1)}_\ell(kr)$ for outgoing waves. As we found in that lecture, the transverse components are given by J9.116, 118

$$\vec{E}^{(M)}_{\ell m} = Z_0 g_\ell(kr) \vec{L} Y_{\ell m}, \qquad \vec{H}^{(M)}_{\ell m} = -\frac{i}{k Z_0} \vec{\nabla} \times \vec{E}^{(M)}_{\ell m} \tag{3}$$

$$\text{or} \quad \vec{H}^{(E)}_{\ell m} = g_\ell(kr) \vec{L} Y_{\ell m}(\theta, \phi), \qquad \vec{E}^{(E)}_{\ell m} = i \frac{Z_0}{k} \vec{\nabla} \times \vec{H}^{(E)}_{\ell m}. \tag{4}$$

We define the vector spherical harmonic functions, for $\ell \geq 1$, as J9.119

$$\vec{X}_{\ell m}(\theta, \phi) := \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} Y_{\ell m}(\theta, \phi),$$

with orthogonality properties J9.120

$$\begin{aligned}
\int d\Omega \vec{X}^*_{\ell'm'}\vec{X}_{\ell m} &= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}}\int d\Omega\Big[\frac{1}{2}\left(L_+^* Y^*_{\ell'm'}\right)(L_+Y_{\ell m}) \\
&\qquad + \frac{1}{2}\left(L_1^*Y^*_{\ell'm'}\right)(L_-Y_{\ell m}) + \left(L_z^*Y^*_{\ell'm'}\right)(L_zY_{\ell m})\Big] \\
&= \int d\Omega\, \frac{Y^*_{\ell'm'}\left[\frac{1}{2}L_-L_+ + \frac{1}{2}L_+L_- + L_z^2\right]Y_{\ell m}}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}} \\
&= \frac{\sqrt{\ell(\ell+1)}}{\sqrt{\ell'(\ell'+1)}}\int d\Omega\, Y^*_{\ell'm'}Y_{\ell m} \\
&= \delta_{\ell\ell'}\delta_{mm'}
\end{aligned}$$

where $\int d\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$, we have used $\vec{L}^{\,2} = \frac{1}{2}L_-L_+ + \frac{1}{2}L_+L_- + L_z^2$, $\int d\Omega(\vec{L}\Phi)^*\Psi = \int d\Omega\Phi^*\vec{L}\Psi$, and $\int d\Omega Y^*_{\ell'm'}Y_{\ell m} = \delta_{\ell\ell'}\delta_{mm'}$. We also have J9.121

$$\begin{aligned}
\int d\Omega \vec{X}^*_{\ell'm'}\cdot\left(\vec{r}\times\vec{X}_{\ell m}\right) &= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}}\int d\Omega\left(\vec{L}^*Y^*_{\ell'm'}\right)\cdot(\vec{r}\times\vec{L})Y_{\ell m} \\
&= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}}\int d\Omega Y^*_{\ell'm'}\vec{L}\cdot(\vec{r}\times\vec{L})Y_{\ell m} \\
&= \frac{1}{\sqrt{\ell(\ell+1)}\sqrt{\ell'(\ell'+1)}}\int d\Omega Y^*_{\ell'm'}\vec{r}\cdot(\vec{L}\times\vec{L})Y_{\ell m} = 0.
\end{aligned}$$

Note in the last equality, it is not because $\vec{L}\times\vec{L}$ vanishes, but because it is $i\vec{L}$, which is perpendicular to $\vec{r}$.

We could reexpress our results for the radiation of arbitrary sources, in terms of the appropriate expansion coefficients a_ℓ multiplying $h_\ell^{(1)}$ for $\vec{r}\cdot\vec{H}$ or $\vec{r}\cdot\vec{E}$. But this is rather messy, and we will skip the rest of the chapter.

2 Scattering

Accelerating charges radiate. We have discussed what a specified oscillating current density will do, but we need to also discuss what current density will be created by an incident electromagnetic field. That is, scattering.

If we have some material which will respond to an incident electromagnetic field, it will also radiate field, generally in all directions. Of most interest is an incident plane wave, and the amplitude for an emitted wave in a particular direction. We will consider small scatterers, of size $\ll \lambda$, and an incident plane wave[1] J10.1

$\vec{n} \rightarrow \hat{r}$?

$$\vec{E}_{\rm inc} = \vec{\epsilon}_i E_i e^{ik\vec{n}_i \cdot \vec{x}}, \qquad \vec{H}_{\rm inc} = \vec{n}_i \times \vec{E}_{\rm inc}/Z_0.$$

where a time dependence $e^{-i\omega t}$ is understood and $k = \omega/c$. If the scatterer is then induced to have electric and magnetic dipole moments $\vec{p}$ and $\vec{m}$ induced by this wave, we will get dipole radiation (for $r \gg \lambda$) as derived earlier sign $\vec{m}$?

$$\vec{E}_{\rm sc} = \frac{k^2 e^{ikr}}{4\pi\epsilon_0 r}\left[(\hat{r} \times \vec{p}) \times \hat{r} + \hat{r} \times \vec{m}/c\right].$$

The crosssectional area of the incident wave that has the energy radiated into a solid angle $d\Omega$ is called the differential cross section. As the flux is given by $\frac{1}{2}\hat{r} \cdot (\vec{E} \times \vec{H}^*)$, and we may wish to examine particular polarizations, projected by $\vec{\epsilon}^* \cdot \vec{E}$, we have the polarized differential cross section is J10.4

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{r}, \vec{\epsilon}; \hat{n}_i, \vec{\epsilon}_i) &= r^2 \frac{\left|\vec{\epsilon}^* \cdot \vec{E}_{\rm sc}\right|^2}{\left|\vec{\epsilon}_i^* \cdot \vec{E}_{\rm inc}\right|^2} \\ &= \frac{k^4}{(4\pi\epsilon_0 E_i)^2}\left|\vec{\epsilon}^* \cdot \vec{p} + (\hat{r} \times \vec{\epsilon}^*) \cdot \vec{m}/c\right|^2. \end{aligned}$$

Because we are assuming the scatterers are small compared to a wavelength, we expect the electric and dipole moment to be quasi-static responses to the applied fields, and thus not dependent on ω, so the overall scattering strength is proportional to $k^4 \propto \omega^4$, which is known as Rayleigh's law.

The responses $\vec{p}$ and $\vec{m}$ can be calculated in simple models. In Jackson §10.1B and §10.1C he does a dielectric sphere and a perfectly conducting sphere. $\perp$ 2/23/09

[1] I am using $\vec{\epsilon}_i$ instead of $\vec{\epsilon}_0$ to remove some of the confusion from using the same symbol as for the permittivity of free space ϵ_0. I will also use i (for incident) more generally instead of 0.